MC

1. **E**

Solution: Since 2021 is an odd integer and $2n$ is always an even integer, then $2021 + 2n$ is always an odd integer, as required.

(Pascal 2017, #10)

2. **D**

Solution: The six angles around the centre of the spinner add to 360°. Thus, $140° + 20° + 4x° = 360°$ or $4x = 360 - 140 - 20 = 200$, and so $x = 50$. Therefore, the sum of the central angles of the shaded regions is $140° + 50° + 50° = 240°$. The probability that the spinner lands on a shaded region is the fraction of the entire central angle that is shaded, which is equal to 240°/360° or 2/3.

(Pascal 2017, #16)

3. **D**

Solution: 711 students like English, so there cannot be more than 711 students who like all three subjects. Since there are more students who like math and more students who like science than those who like English, it is possible that all 711 students who like English also like math and science. Therefore, the maximum possible number of students who like all three subjects is 711, so $x = 711$.

Since 711 students like English and there are 1399 students in total, then $1399 - 711 = 688$ students do not like English. Now, 940 students like science. Since 688 students do not like English, then at most 688 students like science but not english. This means that there are at least $940 - 688 = 252$ students who like both science and English.

Lastly, 1173 students like math. Since there are at least 252 students who like both science and English, then there are at most $1399 - 252 = 1147$ students who either do not like science or do not like English (or both). Since there are 1173 students who like math, then there are at least $1173 - 1147 = 26$ students who like all of math, science, and English. In other words, the minimum possible number of students who like all three subjects is 26, and so $y = 26$.

Since $x = 711$ and $y = 26$, then $x - y = 685$.

(Pascal 2017, #24)

4. **D**

Solution: We label the players P, Q, R, S, T, U. Each player plays 2 games against each of the other 5 players, and so each player plays 10 games. Thus, each player earns between 0 and 10 points, inclusive.

We show that a player must have at least 9.5 points to guarantee that he has more points than every other player. We do this by showing that it is possible to have two players with 9 points, and that if one player has 9.5 or 10 points then every other player has at most 9 points.

Suppose that P and Q each win both of their games against each of R, S, T, and U and tie each of their games against each other. Then P and Q each have a record of 8 wins, 2 ties, 0 losses, giving them each $8 \times 1 + 2 \times 0.5 + 0 \times 0 = 9$ points. We note also that R, S, T, U each have 4 losses (2 against each of P and Q), so have at most 6 points. Therefore, if a player has 9 points, it does not guarantee that they have more points than every other player, since in the scenario above both P and Q have 9 points.

Suppose next that P has 9.5 or 10 points. If P has 10 points, then P won every game that they played, which means that every other player lost at least 2 games, and so can have at most 8 points. If P has 9.5 points, then P must have 9 wins, 1 tie, and 0 losses. (With 9.5 points, P has only "lost" 0.5 points and so cannot have lost any games and can only have tied 1 game). Since P has 9 wins, then P must have beaten each of the other players at least once. (If there was a player that P had not beaten, then P would have at most 4 × 2 = 8 wins.) Since every other player has at least 1 loss, then every other player has at most 9 points. Therefore, if P has 9.5 or 10 points, then P has more points than every other player.

In summary, if a player has 9.5 or 10 points, then they are guaranteed to have more points than every other player, while if they have 9 points, it is possible to have the same number of points as another player.

Thus, the minimum number of points necessary to guarantee having more points than every other player is 9.5.

(Cayley 2015, #22)

5. **E**

Solution: Since m and n are positive integers with $n > 1$ and $m^n = 2^{25} \times 3^{40}$, then 2 and 3 are prime factors of m (since they are prime factors of m^n) and must be the only prime factors of m (since if there were other prime factors of m, then there would be other prime factors of m^n). Therefore, $m = 2^a \times 3^b$ for some positive integers a and b and so $m^n = (2^a \times 3^b)^n = 2^{an} \times 3^{bn}$.

Since $m^n = 2^{25} \times 3^{40}$, then we must have $an = 25$ and $bn = 40$. Since a, b, n are positive integers, then n is a common divisor of 25 and 40. Since $n > 1$, then $n = 5$, which means that $a = 5$ and $b = 8$. In this case, $m = 2^5 \text{ x } 3^8 = 32 \times 6561 = 209\ 952$, which gives $m \times n = 209\ 952 \times 5 = 1\ 049\ 760$.

(Fermat 2017, #20)

Word Problems

1. **23/90**

Solution: The number of two-digit integers in the range 10 to 99 inclusive is 99 – 10 + 1 = 90. We count the number of two-digit positive integers whose tens digit is a multiple of the units (ones) digits.

- If the units digit is 0, there are no possible tens digits.
- If the units digit is 1, the tens digit can be any digit from 1 to 9. This gives 9 such numbers.
- If the units digit is 2, the tens digit can be 2, 4, 6, 8. This gives 4 such numbers.
- If the units digit is 3, the tens digit can be 3, 6, 9. This gives 3 such numbers.
- If the units digit is 4, the tens digit can be 4, 8. This gives 2 such numbers.
- If the units digit is any of 5, 6, 7, 8, 9, the only possibility is that the tens digit equals the units digit. This gives 5 such numbers.

In total, there are 9 + 4 + 3 + 2 + 5 = 23 two-digit positive integers whose tens digit is a multiple of the units digit, and so the probability that a randomly chosen two-digit integer has this property is 23/90.

(CTMC Team 2016, #7)

2. ACB

Solution: Starting with ABC, the magician:

- swaps the first and second cups to obtain BAC, then
- swaps the second and third cups to obtain BCA, then
- swaps the first and third cups to obtain ACB.

The net effect of these 3 moves is to swap the second and third cups. When the magician goes through this sequence of moves 9 times, the net effect starting from ABC is that he swaps the second and third cups 9 times. Since 9 is odd, then the final order is ACB.

(CTMC Team 2016, #4)

3. 1

Solution: Let $x = 1234567$.
Thus, $x - 1 = 1234566$ and $x + 1 = 1234568$.
Therefore,

$$1234567^2 - 1234566 \times 1234568 = x^2 - (x - 1)(x + 1) = x^2 - (x^2 - 1) = 1.$$

(CTMC Team 2019, #12)

Bonus Puzzle

Watch Ted-Ed's "Can you solve the counterfeit riddle?" video for an awesome explanation.

Note: Some questions have been slightly modified from the original.