## Multiple Choice

1. **C**

To maximize the number of songs used, Lauryn should use as many of the shortest length songs as possible. (This is because she can always trade a longer song for a shorter song and shorten the total length used.) If Lauryn uses all 50 songs of 3 minutes in length, this takes 150 minutes. There are 180 − 150 = 30 minutes left, so she can play an additional 30 ÷ 5 = 6 songs that are 5 minutes in length. In total, she plays 50 + 6 = 56 songs.

(Fermat 2009, #8)

2. **E**

When 2 kg of the 10 kg of peanuts are removed, there are 8 kg of peanuts remaining. Since 2 kg of raisins are added, then there are 2 kg of raisins in the bin. The peanuts and raisins are thoroughly mixed. Since 2 kg of this mixture is removed and this is one-fifth of the total mass, then one-fifth of the mass of peanuts (or 8/5 kg) is removed and one-fifth of the mass of raisins (or 2/5 kg) is removed.

This leaves 8 − 8/5 = 32/5 kg of peanuts, and 2 − 2/5 = 8/5 kg of raisins. When 2 kg of raisins are added, the mass of raisins becomes 8/5 + 2 = 18/5 kg. There are 32/5 kg of peanuts and 18/5 kg of raisins in the bin. Therefore, the ratio of the masses is 32/5 : 18/5 = 32 : 18 = 16 : 9.

(Fermat 2014, #21)

3. **B**

The number of points on the circle equals the number of spaces between the points around the circle. Moving from the point labelled 7 to the point labelled 35 requires moving 35 − 7 = 28 points and so 28 spaces around the circle. Since the points labelled 7 and 35 are diametrically opposite, then moving along the circle from 7 to 35 results in travelling halfway around the circle. Since 28 spaces makes half of the circle, then 2 · 28 = 56 spaces make the whole circle. Thus, there are 56 points on the circle, and so n = 56.

(Cayley 2016, #19)

4. **E**

Suppose that Kevin had played $n$ games before his last game. Since he scored an average of 20 points per game over these $n$ games, then he scored $20n$ points over these n games. In his last game, he scored 36 points and so he has now scored $20n + 36$ points in total. But, after his last game, he has now played $n+1$ games and has an average of 21 points scored per game. Therefore, we can also say that his total number of points scored is $21(n + 1)$.

Thus, $21(n + 1) = 20n + 36$ or $21n + 21 = 20n + 36$ and so $n = 15$. This tells us that after 16 games, Kevin has scored 20(15) + 36 = 336 points. For his average to be 22 points per game after 17 games, he must have scored a total of 17 · 22 = 374 points. This would mean that he must score 374 − 336 = 38 points in his next game.

(Cayley 2016, #21)

**5. E**

*Solution:*

Since $x^2 + 3xy + y^2$ = 909 and $3x^2 + xy + 3y^2$ = 1287, then

$$(x^2 + 3xy + y^2) + (3x^2 + xy + 3y^2) = 909 + 1287$$
$$4x^2 + 4xy + 4y^2 = 2196$$
$$x^2 + xy + y^2 = 549$$

Since $x^2 + 3xy + y^2$ = 909 and $x^2 + xy + y^2$ = 549, then

$$(x^2 + 3xy + y^2) - (x^2 + xy + y^2) = 909 - 549$$
$$2xy = 360$$
$$xy = 180$$

Since $x^2 + 3xy + y^2$ = 909 and xy = 180, then

$$(x^2 + 3xy + y^2) - xy = 909 - 180$$
$$x^2 + 2xy + y^2 = 729$$
$$(x + y)^2 = 27^2$$

Therefore, x + y = 27 or x + y = −27. This also shows that x + y cannot equal any of 39, 29, 92, and 41. Therefore, a possible value of x + y is 27.

(Fermat 2020, #22)

## Word Problems

1. We note that 36 000 = 36 × 1000 = $6^2 \times 10^3 = (2 \times 3)^2 \times (2 \times 5)^3 = 2^2 \times 3^2 \times 2^3 \times 5^3 = 2^5 \times 3^2 \times 5^3$. This is called the prime factorization of 36 000. There are many different ways of getting to this factorization, although the final answer will always be the same. Since 36 000 = $2^5 \times 3^2 \times 5^3$ and we want 36 000 = $2^a3^b5^c$ , then a = 5 and b = 2 and c = 3.

Thus, 3a + 4b + 6c = 3(5) + 4(2) + 6(3) = 15 + 8 + 18 = **41**.

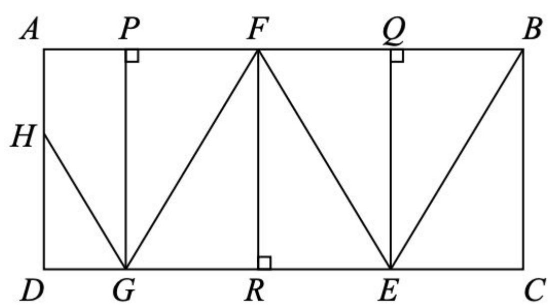
(CIMC 2019, A3)

2. To make *n* as large as possible, we make each of the digits *a*, *b*, *c* as large as possible, starting with *a*. Since *a* is divisible by 2, its largest possible value is *a* = 8, so we try *a* = 8. Consider the two digit integer 8*b*. This integer is a multiple of 3 exactly when *b* = 1, 4, 7. We note that 84 is divisible by 6, but 81 and 87 are not. To make *n* as large possible, we try *b* = 7, which makes *n* = 87*c*. For *n* = 87*c* to be divisible by 5, it must be the case that *c* = 0 or *c* = 5. But 875 = 7 · 125 so 875 is divisible by 7. Therefore, for *n* to be divisible by 5 and not by 7, we choose *c* = 0. Thus, the largest integer *n* that satisfies the given conditions is *n* = **870**.

(CTMC Individual 2018, #5)

3.

**CEMC Solution:** Let points P and Q be on AB so that GP and EQ are perpendicular to AB. Let point R be on DC so that F R is perpendicular to DC.




Note that each of $\triangle$BCE, $\triangle$EQB, $\triangle$EQF, $\triangle$FRE, $\triangle$FRG, and $\triangle$FPG is right-angled and has height equal to BC, which has length 10 m. Since $\angle$BEC = $\angle$FEG, then the angles of $\triangle$BCE and $\triangle$FRE are equal. Since their heights are also equal, these triangles are congruent. Since FREQ and BCEQ are rectangles (each has four right angles) and each is split by its diagonal into two congruent triangles, then $\triangle$EQF and $\triangle$EQB are also congruent to $\triangle$BCE. Similarly, $\triangle$FPG and $\triangle$FRG are congruent to these triangles as well. Let CE = *x* m. Then ER = RG = EC = *x* m.

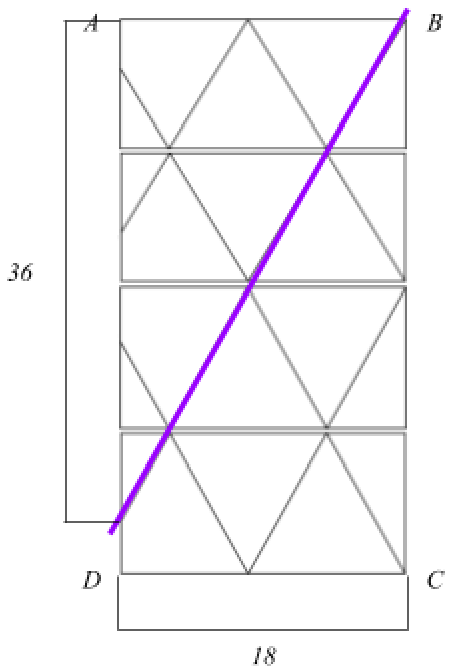
Since $\triangle$HDG is right-angled at D and $\angle$HGD = $\angle$FGE, then $\triangle$HGD is similar to $\triangle$FRG. Since HD : FR = 6 : 10, then DG : RG = 6 : 10 and so DG = $\frac{6}{10}$ RG = $\frac{3}{5}$ *x* m.

Since AB = 18 m and ABCD is a rectangle, then DC = 18 m. But DC = DG + RG + ER + EC = $\frac{3}{5}x + 3x = \frac{18}{5}x$ m. Thus, $\frac{18}{5}x = 18$ and so $x = 5$. Since $x = 5$, then by the Pythagorean Theorem in $\triangle$BCE, BE = $\sqrt{BC^2 + EC^2} = \sqrt{(10m)^2 + (5m)^2} = \sqrt{125m^2} = 5\sqrt{5}$ m.

Now FG = EF = BE = $5\sqrt{5}$ m and GH = $\frac{6}{10}$ FG = $\frac{3}{5}(5\sqrt{5}$ m) = $3\sqrt{5}$ m. Therefore, the length of the path BEFGH is 3($5\sqrt{5}$ m) + ($3\sqrt{5}$ m) or $18\sqrt{5}$ **m**.

(P.S. This solution may be a bit confusing! If so, take a look at this alternative solution!)

**Alternative Solution:** If we were to flip the shape twice on a horizontal axis, and place two on top of one another, we would obtain a long rectangle like so. Thus, the purple line will indicate the length of BEFGH.




We can continue by using the Pythagorean Theorem to solve for the length of the line. When adding four of the rectangles, each with a width of 10 m, we can conclude that (4 x 10) - 4 = 36 m would be the length of the first side. We subtract 4 as in the original rectangle, HD = 6 m, meaning that AH = 4 m. Side AB would remain the same length, 18 m.

Thus, $36^2 + 18^2 = 1620 \rightarrow \sqrt{1620} = 18\sqrt{5}$.

(CTMC Individual 2018, #8)

**Bonus Question:**
This pattern is super simple! Surprisingly, there is no mathematics involved!

**12** read aloud is “one one, one two”. Therefore, we type exactly what that is!
**1112** read aloud is “three one, one two”. Again, we type exactly what is being said.
**3112** read aloud is “one three, two one, one two”. Getting the hang of it yet?

Therefore, the sequence continues with 132112, 1113122112, etc...