MC

1. **D**

Solution: There are 7 days in a week, for a total of 28 days over 4 weeks. Since Tyler sends 5 texts for each day of the weekend, he sends 2*4*5 = 40 texts over the weekends. Over the weekdays, he sends 2 texts everyday, so sends 5*2*4 = 40 texts over the weekdays. In total, he sends 80 texts.

2. **C**

Solution: Taking the prime factorization for each number results in $2^3 * 2^4 \times 3^1 * 3^4 = 2^7 \times 3^5$. Since $6 = 2^1 \times 3^1$, any $6^k = (2^k \times 3^k)$. Thus, the highest possible value of k must be 5, as the smallest power out of $2^7 \times 3^5$ is 5.

(Fermat 2019, #8).

3. **C**

Solution: Since *a, b, and c* are consecutive terms of a geometric sequence, $\frac{c}{b} = \frac{b}{a}$ then $b^2 = ac$. By the quadratic formula, the discriminant of this quadratic equation is $b^2 - 4ac = ac - 4ac = -3ac < 0$. Since a and c are both positive, the discriminant is negative; the parabola does not intersect the x-axis. Furthermore, because the leading coefficient, *a*, is positive, the parabola is entirely above the x-axis.

(Fermat 2003, #20)

4. **B**

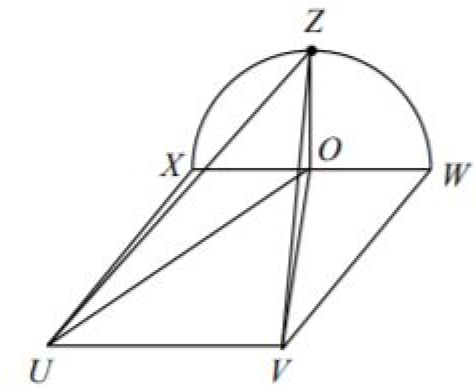
Solution: Placing the three friends on a cartesian plane, we place Tejas on the origin, and Harry on an arbitrary location 10 units away from the origin. For the sake of simplicity, Harry is located on (0,10). Creating an equilateral triangle with points (0,0), (0,10), and the points on the circle x^2 + y^2 = 100, we are able to determine the ratio between the directions in which Cristian will be closer than Harry by measuring the angles of the two triangles, which is 2π/3, or 120 degrees.

The ratio is then established by:
π / (2π/3), or 120/360.

Both of these ratios result in 1/3.

5. **B**

Solution: The perimeter of $UVZ = UV + UZ + VZ$.
We know that $UV = 20$, and we need to calculate UZ *and* VZ.
Let O be the point on XW directly under Z. Since Z is the highest point on the semi-circle, and XW is the diameter, then O is the center of the semi-circle.



Since $UVWX$ is a rectangle, then $XW = UV = 20$, and $UX = VW = 30$.
Since XW is the diameter of the semi-circle, then O is the midpoint of XW and so $XO = WO = 10$.
This means that the radius of the semicircle is 10, and so $OZ = 10$.
Triangles UXO and VWO are both right-angled, since $UVWX$ is a rectangle.
By the Pythagorean theorem, $UO^2 = UX^2 + XO^2 = 30^2 + 10^2 = 900 + 100 = 1000$, *and*
$VO^2 = VO^2 + OZ^2$
Since $UO^2 = VO^2 = 1000$, *then* $UZ^2 = VZ^2 = 1000 + 10^2 = 1100$, *or* $UZ = VZ = \sqrt{1100}$
Therefore, the perimeter of triangle *UVZi is* $20 + 2\sqrt{1100}$, which is about 86.332.
Of the given choices, the closest is 86.

(Pascal 2017, #22)

Word Problems

1. **Max 5, Min 1**

We rewrite by completing the square as $f(x) = \sin^2 x - 2\sin x + 2 = (\sin x - 1)^2 + 1$.
Therefore, since $(\sin x - 1)^2 \geq 0$, then $f(x) \geq 1$, and in fact $f(x) = 1$ when $\sin x = 1$ (which occurs for instance when $x = 90$ degrees.).
Thus, the minimum value of $f(x)$ is 1.
To maximize $f(x)$, we must maximize $(\sin x - 1)^2$. Since $-1 \leq \sin x \leq 1$, then $(\sin x - 1)2$ is maximized when $\sin x = -1$ (for instance, when $x = 270$ degrees). In this case, $(\sin x - 1)^2 = 4$, so $f(x) = 5$. Thus, the maximum value of $f(x)$ is 5.

2. **66.**

Solution We recall that a positive integer is divisible by 3 whenever the sum of its digits is divisible by 3. Since the sum of the digits does not depend on the order of the digits, then rearranging the digits of a positive integer that is divisible by 3 produces another positive integer that is divisible by 3.

Note that 10 000 is not divisible by 3. Every other positive integer between 1000 and 10 000 is a four-digit integer. Consider a four-digit positive integer whose four digits are consecutive integers. We can rearrange the digits of this integer in decreasing order to obtain one of the positive integers 3210, 4321, 5432, 6543, 7654, 8765, and 9876. The sums of the digits of these integers are 6, 10, 14, 18, 22, 26, and 30, respectively. Of these, 6, 18 and 30 are the only sums that are divisible by 3, so 3210, 6543 and 9876 are the only ones divisible by 3.

Since rearranging the digits does not affect whether an integer is divisible by 3, then a four-digit integer satisfies the given conditions if its digits are rearrangements of 3210 or 6543 or 9876.

There are 24 four-digit integers whose digits are rearrangements of 6543. (There are 4 possibilities for the thousands digit, then 3 possibilities for the hundreds digit, then 2 possibilities for the tens digit, and 1 possibility for the units digit, and so $4 \times 3 \times 2 \times 1 = 24$ integers that use these digits.)

Similarly, there are 24 four-digit integers whose digits are rearrangements of 9876. Finally, there are 18 four-digit integers whose digits are rearrangements of 3210. (There are 3 possibilities for the thousands digits (because 0 cannot be the thousands digit), then 3 possibilities for the hundreds digit, 2 for the tens digit, and 1 for the units digit, and so $3\times3\times2\times1 = 18$ integers that use these digits.) In total, there are $24 + 24 + 18 = 66$ positive integers with these three properties.

(CSMC 2013, #5)

3. **55, 29, 15350.**

A) Solution 1: Manually draw and count.

Solution 2: This pattern can be written as a recursive series. The first few terms of the series are 5, 13, 24… So, using Gauss' famous 1-100 counting strategy, we can determine that the series will be $S(n) = n(10+3(n-1)) / 2$. $S(n) = 55$.

B) Solution: Using S(n) from above, we can determine that $S(9) - S(8) = 29$.

Solution 2: We calculate the difference between these ink lengths by mentally starting with Figure 8 and determining what needs to be added to Figure 8 to produce Figure 9. Figure 9 is a copy of Figure 8 with a regular pentagon of side length 9 added and overlapping segments removed.

> A regular pentagon with side length 9 has an ink length of $5 \cdot 9 = 45$.
> The overlapping segments that must be removed are the two edges coming out of T in Figure 8, which each have length 8.
> Therefore, the difference in ink length is $5 \cdot 9 - 2 \cdot 8 = 45 - 16 = 29$.

C) Solution: Using S(n), we can determine that $S(100) = 15350$.

The ink length of Figure 100 can be calculated by starting with Figure 99 and determining how much ink length needs to be added.

Similarly, the ink length of Figure 99 can be calculated by starting with Figure 98 and determining how much ink length needs to be added.

In general, the ink length of Figure k can be calculated by starting with Figure $(k - 1)$ and determining how much ink length needs to be added.

Using this process, we can start with Figure 1, add to this to get Figure 2, add to this to get Figure 3, and so on, all of the way up to Figure 100.

To obtain the difference in ink length between Figure k and Figure $(k - 1)$, we can model what we did in (b): We calculate the difference between these ink lengths by mentally starting with Figure $(k - 1)$ and determining what needs to be added to Figure $(k - 1)$ to produce Figure k. Figure k is a copy of Figure $(k-1)$ with a regular pentagon of side length k added and overlapping segments removed.

A regular pentagon with side length k has an ink length of $5k$.The overlap segments that must be removed are the two edges coming out of T in Figure $(k - 1)$, which each have length $(k - 1)$.

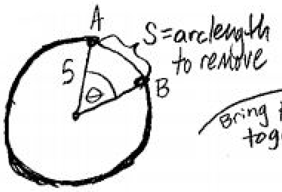
Therefore, the difference in ink length is $5k - 2(k - 1) = 3k + 2$.

Therefore, to get from the ink length of Figure 1 (which is 5) to the ink length of Figure 100 we need to add each of the differences $3k+2$ (calculated above) from $k = 2$ to $k = 100$. Therefore, the ink length of Figure 100 is

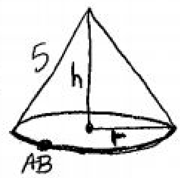
$$
\begin{aligned}
&5 + (3(2) + 2) + (3(3) + 2) + \cdots + (3(99) + 2) + (3(100) + 2) \\
&= (3(1) + 2) + (3(2) + 2) + (3(3) + 2) + \cdots + (3(99) + 2) + (3(100) + 2) \\
&= 3(1 + 2 + 3 + \cdots + 99 + 100) + 100(2) \\
&= 3\left(\frac{1}{2}(100)(101)\right) + 200 \\
&= 150(101) + 200 \\
&= 15150 + 200 \\
&= 15350
\end{aligned}
$$

Therefore, the ink length of Figure 100 is 15 350.

Challenge:



Bring points A&B together



Let C = circumference

Arc length formula: $S = r\theta$; θ in radians

$h^2 + r^2 = 5^2$

$h = \sqrt{25 - r^2}$, but $r = 5 - \frac{5}{2\pi}\theta$

so $\boxed{h = \sqrt{25 - \left(5 - \frac{5}{2\pi}\theta\right)^2}}$

$C_{Disk} - S = C_{Base\ of\ cone}$

$10\pi - 5\theta = 2\pi r$

so $10\pi - 5\theta = 2\pi r$

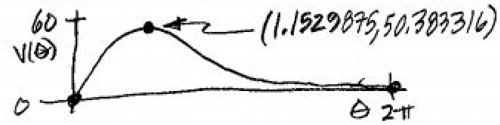
$r = \frac{10\pi}{2\pi} - \frac{5\theta}{2\pi}$

$\boxed{r = 5 - \frac{5}{2\pi}\theta}$

$V_{cone} = \frac{\pi}{3} r^2 h$

$V(\theta) = \frac{\pi}{3}\left(5 - \frac{5}{2\pi}\theta\right)^2 \sqrt{25 - \left(5 - \frac{5}{2\pi}\theta\right)^2}$, $\theta \in [0, 2\pi]$

* graph this on calculator on window $X[0, 2\pi]$, $Y[0, 60]$, use $\boxed{2nd}$ $\boxed{trace}$ $\boxed{\#5}$



So optimal central angle in degrees:

$(1.1529\ldots \text{ rad})\left(\frac{180^\circ}{\pi \text{ rad}}\right) \approx \boxed{66.061^\circ}$