MC

1. **E**

Solution: The equation simplifies to $|x| - 1 = \pm 1$, i.e., $|x| = 0$ or 2. The former gives $x = 0$ while the latter gives $x = \pm 2$, so the answer is $2 - (-2) = 4$.

2. **B**

Solution 1: The first condition says the number of pizzas can be (in increasing order) 1/3, 5/6, 4/3, 11/6, etc.
The second condition says the number of pizzas can be (in increasing order) 1/6, 1/2, 5/6, 7/6, 3/2, 11/6, etc.
So the answer is 5/6.

Solution 2:
If Mathlete eats a ½-pizzas and has a ⅓-pizza left, then $a/2 + 1/3 = x$, the total number of pizzas. This gives $3a + 2 = 6x$, i.e., $3(a + 1) = 6x + 1$.
If Mathlete eats b ⅓-pizzas and has a ⅙-pizza left, then $b/3 + 1/6 = x$. This gives $2b + 1 = 6x$, i.e., $2(b + 1) = 6x + 1$.
We now have $3(a + 1) = 2(b + 1) = 6x + 1$, which must be a positive integer that is both a multiple of 3 and a multiple of 2. The smallest possible value of $6x + 1$ is $\gcd(3, 2) = 6$, giving a minimum x value of 5/6.

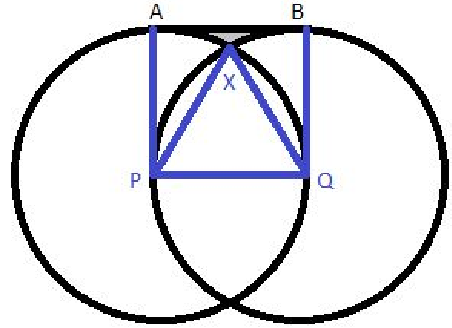
3. **C**

Solution:
Answer = (probability of being in B and getting slime box) / (probability of getting slime box)
$= (\frac{1}{3} \times \frac{1}{2}) / (\frac{1}{3} \times \frac{1}{2} + \frac{1}{3})$
$= \frac{1}{3}$.

Remark: It is important to remember that $P(A \cap B) = P(A)\,P(B \mid A)$ and its equivalent forms. (The equation reads "the probability that A and B both happen equals the probability that A happens times the probability that B happens given A happens.) In this question, we used the variant $P(B \mid A) = P(A \cap B) / P(A)$, where A represents "getting a slime box" and "B" represents "being in room B."

4. **E**

Solution:
As shown, let the centers of the two circles be P and Q, and let one of the intersections be X.
Note that PQX is an equilateral triangle and $\angle APX = \angle BQX = 30°$ (a twelfth of a circle).



Area of shaded region = (Area of *ABQP*) - (Area of sector *APX*) - (Area of sector *BQX*) - (Area of *PQX*)
$= 1^2 - \pi/12 - \pi/12 - 1^2 \times \text{sqrt}(3)/4$
$= 1 - \pi/6 - \text{sqrt}(3)/4$
≈ 0.0434.

5. **B**

Solution 1:
$x^2 + 2y^2 = 2xy$
$x^2 - 2xy + y^2 + y^2 = 0$
$(x - y)^2 + y^2 = 0$.
Since $(x - y)^2$ and y^2 are both non-negative, they are both zero. Therefore, $x = y$ and $y = 0$. So $(x, y) = (0, 0)$. There's exactly one solution pair.

Solution 2:
If $y = 0$, then $x^2 = 0$, so $x = 0$. This gives one solution $(x, y) = (0, 0)$.
If $y \neq 0$, then the original equation becomes

$$(x/y)^2 + 2 = 2(x/y)$$
$$(x/y)^2 - 2(x/y) + 2 = 0.$$

The discriminant of this quadratic in (x/y) is $(-2)^2 - 4(1)(2) = -4 < 0$, so there is no real solution in (x/y) and hence no real solution in (x, y) in this case.

To conclude, there is exactly one solution pair $(x, y) = (0, 0)$.

Word Problems

1. **Yes.**

Solution:
See the figure (from Jonathan Ge) for the construction.

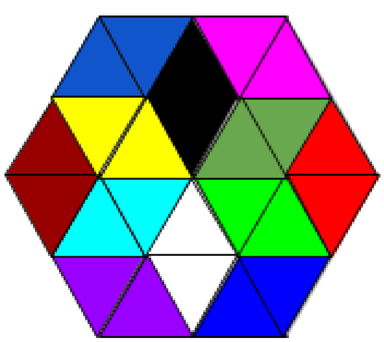


Fig. 1

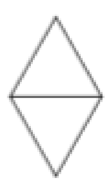
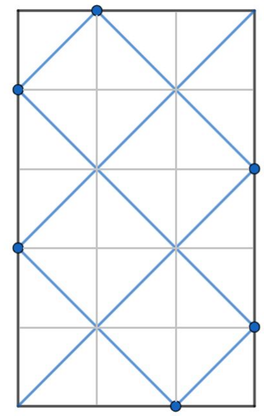


Fig. 2

2. 6



Solution 1: As shown in the diagram, the billiard ball bounces 6 times before entering a corner pocket.

Solution 2: Solve the bonus question first. Then the answer is just $(5 + 3)/\gcd(5, 3) - 2 = 8 - 2 = 6$.

Answer to bonus question: $(m + n)/\gcd(m, n) - 2$
Solution to bonus question:
<u>Note: The question didn't say whether there are edge pockets that the ball may enter before it arrives at a corner pocket. In this solution, we'll disregard the edge pockets.</u>

Let m be the distance between the two vertical edges and n the distance between the two horizontal edges.

Consider the bounces off the vertical edges and those off the horizontal edges separately.

Note that if there are x bounces off the vertical edges, then the ball traversed the width of the table $(x + 1)$ times: from the start to the 1st vertical-edge bounce (1 traversal), between any two consecutive vertical-edge bounces ($(x - 1)$ traversals), and from the last vertical-edge bounce to the corner pocket (1 traversal). Therefore, since the width of the table is m, the total horizontal distance traveled is

$$m(x + 1).$$

Similarly, if there are y bounces off the horizontal edges, then the ball traversed the height of the table $(y + 1)$ times and the total vertical distance traveled is

$$n(y + 1).$$

Because the horizontal distance and vertical distance traveled are equal between any two consecutive bounces (or between the start and the first bounce, or between the last bounce and the corner pocket), the total horizontal distance traveled from the start to the final corner pocket is equal to the total vertical distance traveled. In other words,

$$m(x + 1) = n(y + 1) := d.$$

To minimize $(x + y)$, it is sufficient to minimize $(x + 1)$ and $(y + 1)$ at the same time. When they are both minimized, d is minimized. Since $m \mid d$ and $n \mid d$, the minimum of d is $\operatorname{lcm}(m, n)$. In this case,

$$x + y = (x + 1) + (y + 1) - 2 = d/m + d/n - 2 = \operatorname{lcm}(m, n)/m + \operatorname{lcm}(m, n)/n - 2$$
$$= \operatorname{lcm}(m, n)(1/m + 1/n) - 2 = \operatorname{lcm}(m, n)(m + n)/(mn) - 2 = (m + n)/\gcd(m, n) - 2.$$

3. -1

Solution 1:
$x^2 + 1/x^2 = (x + 1/x)^2 - 2 = 1^2 - 2 = -1.$

Solution 2:
$x + 1/x = 1$
$(x + 1/x)^2 = 1^2$

$x^2 + 2 + 1/x^2 = 1$
$x^2 + 1/x^2 = -1$.

"Fun" fact:
The two solutions to x are $-\omega$ and $-\omega^2$, where $\omega = (-1 + \text{sqrt}(3)i)/2 = e^{2\pi i/3}$.