MC

1. **B**

Solution: The volume of the cuboid is 2 × 4 x 8 = 64. We see that $4^3 = 64$ so the edge length of the cube is 4.

(Pascal 2006, #5)

2. **D**

Solution: Let the hypotenuse be c and the legs be a and b. By the Pythagorean theorem, $a^2 + b^2 = c^2$ so $1800 = a^2 + b^2 + c^2 = 2c^2$. Solving for c gives $c = \sqrt{(1800 / 2)} = \sqrt{900} = 30$.

(Cayley 2006, #18)

3. **D**

Solution: If ♥ is greater than 2, then its cube would be greater than 20. If ♥ = 1, then ▽ = 1 which is not distinct. Thus, ♥ = 2 and ▽ = 8 so ▽ × ▽ = 64.

(Fermat 2018, #5)

4. **B**

Solution: 4Y means four yellow marbles, etc. So we have 8Y, 7R, 5B in the bag.

First we show that 8 fails. Taking 5R and 3B leaves 8Y, 2R, 2B in the bag which does not meet the criteria.

There are two ways of failing:

1. leaving less than 3 marbles of two colours (6Y, 1R, 2B fails)
2. leaving less than 4 marbles of each colour (3Y, 2R, 3B fails)

Every other configuration passes.

Let's try to fail by taking at most 7 marbles using the first way. Leaving less than 3 yellow, red and black marbles requires taking 6Y, 5R, 3R respectively. Adding any two of these exceeds 7 marbles so this is not possible.

Let's try the second way. Leaving less than 4 yellow, red and black marbles requires taking 5Y, 4R, 2R respectively. These add to taking 11 marbles in total which is not possible.

Thus, it is impossible to fail by taking 7 marbles or less.

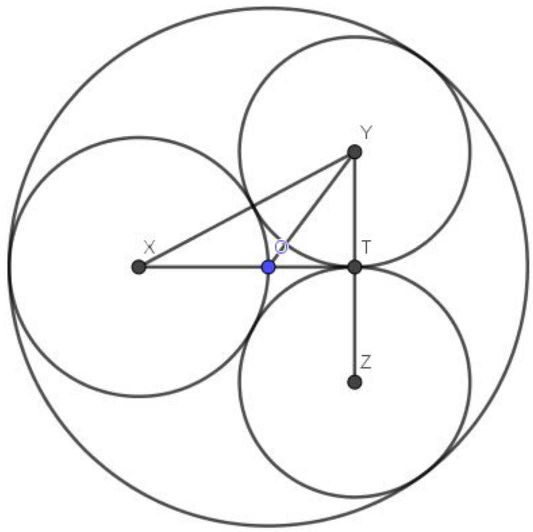
(Fermat 2006, #20)

5. **E**

Solution: Let O be the center of the largest circle. Since O is on the circle centered at X, the radius of the largest circle is 2. Let r be the radius of the small circles.

Notice that by symmetry, the circles centered at Y and Z are tangent to the line XO at T.

We find that $XY = r + 1$, $XO = 1$, $OY = 2 - r$, $YT = r$.



Since YT is perpendicular to XO,

$$OT^2 = OY^2 - YT^2 = (2 - r)^2 - r^2 = 4\,(1 - r).$$

We also see that,

$$XY^2 = YT^2 + XT^2$$
$$(r + 1)^2 = r^2 + (XO + OT)^2$$
$$2r + 1 = 1 + 2\,OT + OT^2$$
$$2r = 2\,OT + 4\,(1 - r)$$
$$3r - 2 = OT$$

Thus,

$$4\,(1 - r) = (3r - 2)^2$$
$$4 - 4r = 9r^2 - 12r + 4$$
$$0 = 9r^2 - 8r$$
$$0 = r\,(9r - 8)$$

but since $r \neq 0$, $r = 8/9$ which is closest to 0.89.

(Fermat 2020, #24)

Word Problems

1. **4**

Solution: From $12x = 4y + 2$ we see that $6x = 2y + 1$.

Multiplying this by 3 gives $18x = 6y + 3$ so $6y - 18x = -3$.

Thus, $6y - 18x + 7 = -3 + 7 = 4$.

(CIMC 2015, A4)

2. **3/4**

Solution: Using 2 with $x = 0$ gives $f(0) = f(0)/2$ so $f(0) = 0$.

Using 1 with $x = 0$ gives $f(1) = 1 - f(0) = 1$.

Using 2 with $x = 1$ gives $f(1/3) = f(1)/2 = 1/2$.

Using 1 with $x = 1/3$ gives $f(2/3) = 1 - f(1/3) = 1/2$.

Using 2 with $x = 1/3$ gives $f(1/9) = f(1/3)/2 = 1/4$.
Using 1 with $x = 1/9$ gives $f(8/9) = 1 - f(1/9) = 3/4$.

Using 2 with $x = 2/3$ gives $f(2/9) = f(2/3)/2 = 1/4$.
Using 1 with $x = 2/9$ gives $f(7/9) = 1 - f(2/9) = 3/4$.

Using 3, we find that for all real numbers t between 7/9 and 8/9, $f(t) = 3/4$. (Why?)
Since $7/9 \leq 6/7 \leq 8/9$, $f(6/7) = 3/4$.

Note: f is the Cantor function

(CSMC 2020, A6)

3.

Solution:

1. **add**(one, two) = **add**(one, **succ**(one))
   = **succ**(**add**(one, one))
   = **succ**(**add**(one, **succ**(zero)))
   = **succ**(**succ**(**add**(one, zero)))
   = **succ**(**succ**(one)))
   = three

   All of the equal signs are true by *definition*.

2. **add**(zero, two) = **add**(zero, two)
   = **add**(zero, **succ**(one))
   = **succ**(**add**(zero, one))
   = **succ**(**add**(zero, **succ**(zero)))
   = **succ**(**succ**(**add**(zero, zero)))
   = **succ**(**succ**(zero))
   = two

   Again, all of the equal signs are true by *definition*.

3. Induct on n.
   a. n = zero

      add(zero, n) = **add**(zero, zero)
      = zero
      = n

   b. Now suppose **add**(zero, n) = n.
      We want to show that **add**(zero, **succ**(n)) = **succ**(n).

      add(zero, **succ**(n)) = **succ**(**add**(zero, n))
      = **succ**(n)

We replaced the **add**(zero, n) in the equation with n.

Note: This question gives a beginner's introduction to the Peano axioms.

Note: Some questions have been slightly modified from the original.