Second SJAM Mathematics Olympiad—Solutions

Dec. 21, 2020 – Jan. 17, 2021

1. [1 pt] Given that $x - 3y + 3z = 7$ and $2x + 4y + z = 9$, determine the value of $x + y + z$.

(Problem by Zed)

Solution:

$$(x - 3y + 3z) + 2(2x + 4y + z) = 7 + 2(9) = 25$$

. . . . [0.5 pts]

$$\begin{aligned} 5x + 5y + 5z &= 25 \\ x + y + z &= 5. \end{aligned}$$

. . . . [0.5 pts]

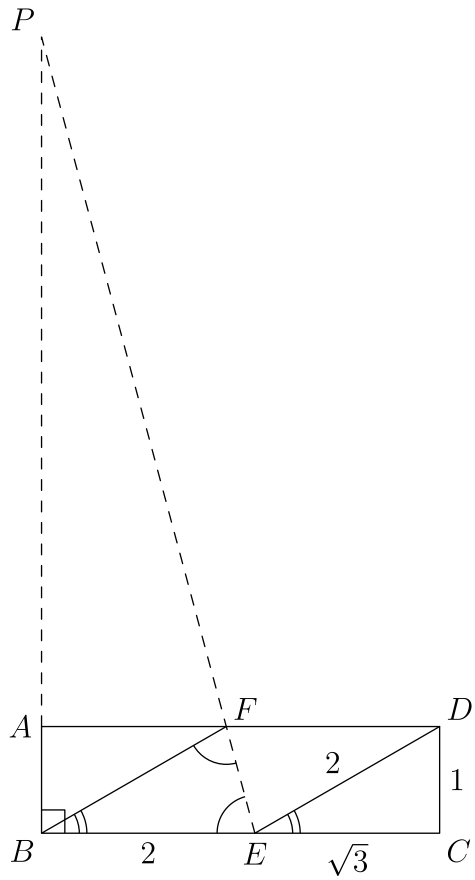
(Solution by Zed)

Remark: We can also solve for two of the variables (e.g. y, z) in terms of the other variable (e.g. x).

2. [2 pts] Let $ABCD$ be a rectangle. E is on side BC and F is on side DA such that $BEDF$ is a rhombus. If $BE = 2$ and $CE = \sqrt{3}$, determine the size of the acute angle between lines AB and EF.

(Problem by Zed)

Solution:



$BEDF$ is a rhombus, so $DE = BE = 2$. Along with $CE = \sqrt{3}$, this shows that $\triangle EDC$ is a 30-60-90 triangle with $\angle CED = 30°$. Since $BF \parallel ED$, we have $\angle EBF = 30°$.

.... [1 pt]

The fact that $BEDF$ is a rhombus also means $BE = BF$, so $\angle BEF$ and $\angle BFE$ are equal and add up to $180° - \angle EBF = 150°$. Therefore, $\angle BEF = \frac{1}{2}(150°) = 75°$.

.... [0.5 pts]

Let P be the intersection of AB and EF. Then the acute angle between AB and EF is $\angle P = 180° - \angle PBE - \angle BEF = 180° - 90° - 75° = 15°$. .... [0.5 pts]

(Solution by Zed)

3. [3 pts] Biscuit is buying 656 biscuits for his boss, Tian. Biscuit needs to get Tian precisely 656 biscuits, no more, no less; otherwise, Biscuit will be fired. Unfortunately, all biscuits in the world come in packs of 7 or 13. How many packs of each kind does Biscuit need to buy to have exactly 656 biscuits for Tian?

(Problem by Biscuit)

Solution:

Say Biscuit buys x packs of 7 biscuits and y packs of 13 biscuits, where x and y are non-negative integers. Then, we need to solve the equation

$$7x + 13y = 656.$$

Note that $7(90) + 13(2) = 656$, so we can write the original equation as

$$\begin{aligned} 7x + 13y &= 7(90) + 13(2) \\ 7(x - 90) + 13(y - 2) &= 0 \\ 13(y - 2) &= 7(90 - x). \end{aligned}$$

.... [1 pt]

Since 13 and 7 are relatively prime, we must have $y - 2$ is divisible by 7 and $90 - x$ is divisible by 13. So let $y - 2 = 7k$ for integer k. Then $y = 7k + 2$. Plugging this into the equation above gives

$$\begin{aligned} 13(7k) &= 7(90 - x) \\ 13k &= 90 - x \\ x &= 90 - 13k. \end{aligned}$$

.... [1 pt]

In conclusion, $(x, y) = (90 - 13k, 7k + 2)$ where $x, y \geq 0$, so k must satisfy $0 \leq k \leq 6$.

.... [1 pt]

Explicitly writing out all the possibilities, we have

k	# of packs of 7	# of packs of 13
0	90	2
1	77	9
2	64	16
3	51	23
4	38	30
5	25	37
6	12	44

(Solution by Zed)

4. [4 pts] The equation

$$x^3 - 3kx^2 + (3k^2 + p)x - k^3 - pk = 0,$$

when solved for x, gives three distinct real roots. If two of these roots are a and b ($a \neq b$), find all possible pairs (p, k). Write your answers in terms of a and b.

(Problem by Biscuit and Zed)

Solution:

$$\begin{aligned}
x^3 - 3kx^2 + (3k^2 + p)x - k^3 - pk &= 0 \\
\left(x^3 - 3kx^2 + 3k^2x - k^3\right) + (px - pk) &= 0 \\
(x-k)^3 + p(x-k) &= 0 \\
(x-k)\left((x-k)^2 + p\right) &= 0.
\end{aligned}$$

.... [1.5 pts]

Therefore, the three distinct real roots in x are

$$x = k, k \pm \sqrt{-p}.$$

.... [0.5 pts]

If $a < b$, then there are three cases:

i) $a = k - \sqrt{-p}$ and $b = k$. This gives $k = b$ and $p = -(b-a)^2$.

ii) $a = k - \sqrt{-p}$ and $b = k + \sqrt{-p}$. This gives $k = \frac{a+b}{2}$ and $p = -\left(\frac{b-a}{2}\right)^2$.

iii) $a = k$ and $b = k + \sqrt{-p}$. This gives $k = a$ and $p = -(b-a)^2$.

.... [1.5 pts]

The $b < a$ case gives the same three solutions as above but with a and b swapped, i.e., $(k, p) = (a, -(a-b)^2)$, $\left(\frac{b+a}{2}, -\left(\frac{a-b}{2}\right)^2\right)$, $(b, -(a-b)^2)$. These happen to be identical to the three solutions obtained for the $a < b$ case.

.... [0.5 pts]

In conclusion, we have three solutions to (p, k):

$$(p, k) = \left(-(a-b)^2, a\right), \left(-\left(\frac{a-b}{2}\right)^2, \frac{a+b}{2}\right), \left(-(a-b)^2, b\right).$$

(Solution by Zed)

5. [5 pts] Show that 8 is the minimum number of cards Cookie needs to draw from a standard 52-card deck to guarantee that he has two cards where the value on one of them is a multiple of the value on the other card. (Assume that A = 1, J = 11, Q = 12, and K = 13.)

(Problem by Zed)

Solution:

If Cookie only chose 7 cards or fewer, then choosing cards with distinct numbers from the set $\{7, 8, 9, 10, 11, 12, 13\}$ will allow him to have no card that is a multiple of another card. . . . . [1 pt]

This is because the ratio of any card value to any smaller card value would be greater than 1 and at most $13/7 < 2$, meaning that this ratio cannot be an integer.

. . . . [0.5 pts]

If Cookie chose 8 cards, then consider the set $S = \{1, 2, 3, \ldots, 13\}$. Partition S into 7 mutually disjoint subsets $\{1, 13\}, \{2, 4, 8\}, \{3, 9\}, \{5, 10\}, \{6, 12\}, \{7\}$, $\{11\}$. Then by the pigeonhole principle, Cookie must have chosen two numbers from the same subset. One of these two number must be a multiple of the other. . . . . [3.5 pts]

In conclusion, 8 is the minimum number of cards Cookie has to choose such that one card is a multiple of another.

(Solution by Zed)

Remark: In general, if Cookie chose k (not necessarily distinct) numbers from the set $\{1, 2, \ldots, n\}$, then the minimum k that would guarantee one chosen number to be a multiple of another is $\lceil \frac{n}{2} \rceil + 1$, where $\lceil x \rceil$ denotes the least integer not less than x. Can you prove it? You may check this out for some inspiration.

6. [6 pts] Every point of the plane is colored one of two colors: gamboge and razzmatazz. Prove that, for all $d > 0$, there exists a monochromatic isosceles triangle with positive area and leg length d.

   Note: A *monochromatic* triangle is one whose vertices are of the same color.

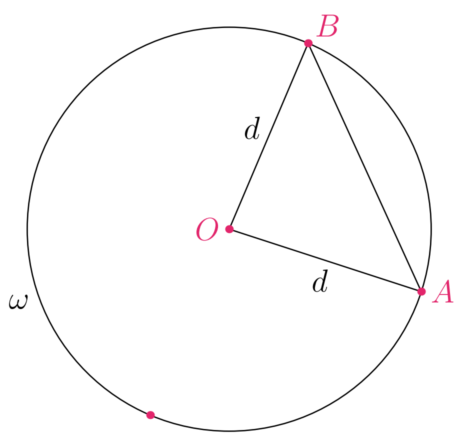
   (Problem by Zed)

   Solution:

   Consider any circle ω with center O and radius d. WLOG, let O be razzmatazz.

   .... [0.5 pts]

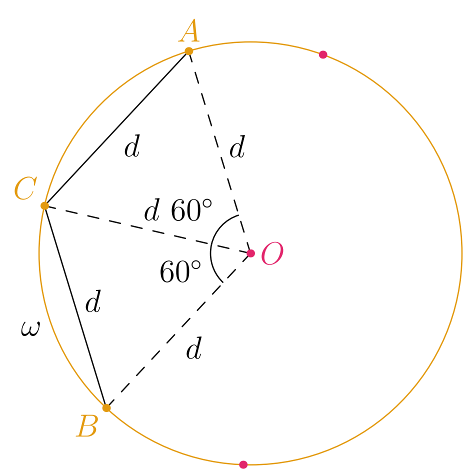
   Case 1:



   If there are three distinct points on ω that are razzmatazz, then two of them (call them A and B) are not endpoints of a diameter of ω. Now, $OA = OB$, so $\triangle OAB$ is a non-degenerate isosceles triangle with razzmatazz vertices and leg length d.

   .... [2.5 pts]

   Case 2:



If there are at most two points on ω is razzmatazz, then there exists a completely gamboge arc of ω with gamboge endpoints A and B and central angle 120°. If C is the midpoint of the arc, then $CA = CB$, so $\triangle CAB$ is a non-degenerate isosceles triangle with gamboge vertices. ....[2 pts]

Since $\triangle OAC$ satisfies $OA = OC$ and $\angle AOC = 120^\circ/2 = 60^\circ$, it is an equilateral triangle. Therefore, $\triangle CAB$ has leg length $CA = OA = d$. ....[1 pt]

In conclusion, it is always possible to find a non-degenerate isosceles triangle with vertices of the same color and leg length d.

(Solution by Zed)

7. [7 pts] Find the maximum possible value for positive integer n such that the following property holds: there exists some positive integer M such that p^2-1 is divisible by n for all prime numbers $p \geq M$.

(Problem by Zed)

Solution:

We claim that the answer is $\boxed{24}$.

When $n=24$, consider $M=5$. Then $2 \nmid p$ and $3 \nmid p$.[1]

Since p is odd, $p \equiv \pm 1, \pm 3 \pmod 8$.[2] But $(\pm 1)^2 \equiv (\pm 3)^2 \equiv 1 \pmod 8$, so $p^2 \equiv 1 \pmod 8$.[3] .... [1 pt]

Also, $p \equiv \pm 1 \pmod 3$, meaning $p^2 \equiv 1 \pmod 3$. .... [0.5 pts]

We have therefore proven that p^2-1 is divisible by both 3 and 8. Since $\gcd(3,8)=1$, we get that p^2-1 is divisible by $3 \times 8 = 24$ for all primes $p \geq 5$. .... [0.5 pts]

Now, assume that positive integer M satisfies p^2-1 is divisible by n for all prime numbers $p \geq M$. We will show that $n \leq 24$ in three steps.

First, we show that if prime $q \mid n$, then $q=2$ or 3. If q is even, then $q=2$. If q is odd, then there exists a prime $p = kq+2 \geq M$ for some integer k (Dirichlet's theorem). Since $p \equiv 2 \pmod q$, we have $p^2-1 \equiv 2^2-1 \equiv 3 \pmod q$. But if $q \mid n$, then $p^2-1 \equiv 0 \pmod q$, so $q \mid 3$, i.e., $q=3$. We have now proven that the only prime factors of n are 2 and/or 3. .... [2 pts]

Next, we show that if $2^\alpha \mid n$ for integer α, then $\alpha \leq 3$. Note that there exists a prime $p = 2^\alpha k + 3 \geq M$ for some integer k (Dirichlet's theorem). Since $p \equiv 3 \pmod{2^\alpha}$, we have $p^2-1 \equiv 3^2-1 \equiv 8 \pmod{2^\alpha}$. But if $2^\alpha \mid n$, then $p^2-1 \equiv 0 \pmod{2^\alpha}$, so $2^\alpha \mid 8$, i.e., $\alpha \leq 3$. .... [1.5 pts]

Finally, we show that if $3^\beta \mid n$ for integer β, then $\beta \leq 1$. Note that there exists a prime $p = 3^\beta k + 2 \geq M$ for some integer k (Dirichlet's theorem). Since $p \equiv 2 \pmod{3^\beta}$, we have $p^2-1 \equiv 2^2-1 \equiv 3 \pmod{3^\beta}$. But if $3^\beta \mid n$, then $p^2-1 \equiv 0 \pmod{3^\beta}$, so $3^\beta \mid 3$, i.e., $\beta \leq 1$. .... [1.5 pts]

From the argument above, the prime factorization of n must be of the form $n = 2^\alpha \cdot 3^\beta$, where $\alpha \leq 3$ and $\beta \leq 1$. Hence, $n \leq 2^3 \cdot 3^1 = 24$.

In conclusion, the largest possible value for n is 24.

(Solution by Zed)

[1] $a \nmid b$ means b is not divisible by a; $a \mid b$ means b is divisible by a.

[2] $a \equiv b \pmod m$ means a and b give the same remainder when divided by m.

[3] Here, we used the fact that if $a \equiv b \pmod m$, then $a^2 \equiv b^2 \pmod m$.

8. [8 pts] If $n^2+6n+11=x^2+y^2$ for positive integers n, x and y, prove that $\left\lfloor \frac{xy}{2} \right\rfloor$ is composite.

Note: $\lfloor t \rfloor$ represents the greatest integer less than or equal to t.

(Problem by Tian)

Solution:

Notice that $n^2+6n+11=(n+3)^2+2 \equiv 2,3 \pmod 4$.[4] But $x^2+y^2 \equiv 0,1,2 \pmod 4$, so $(n+3)^2+2 \equiv x^2+y^2 \equiv 2 \pmod 4$. Therefore, $n+3$ is even, meaning n is odd, and x, y must also be odd. . . . . [2 pts]

Hence, let $x=2a+1$, $y=2b+1$, $n=2k+1$, where a, b, k are non-negative integers. Substituting these into $(n+3)^2+2=x^2+y^2$ gives

$$\begin{aligned}
(2k+4)^2+2 &= (2a+1)^2+(2b+1)^2 \\
4(k+2)^2+2 &= 4a^2+4a+1+4b^2+4b+1 \\
(k+2)^2 &= a^2+a+b^2+b \qquad (*) \\
(k+2)^2 &= (a-b)^2+2ab+a+b.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\lfloor \frac{xy}{2} \right\rfloor &= \left\lfloor \frac{(2a+1)(2b+1)}{2} \right\rfloor \\
&= \left\lfloor \frac{4ab+2a+2b+1}{2} \right\rfloor \\
&= 2ab+a+b \\
&= (k+2)^2-(a-b)^2 \\
&= (k+2+a-b)(k+2-a+b)
\end{aligned}$$

. . . . [3 pts]

It now suffices to show that both factors $k+2+a-b$ and $k+2-a+b$ are greater than 1. WLOG, let $a \geq b$, which gives $k+2+a-b>1$. . . . . [1 pt]

If $x=y=1$, then $n^2+6n=x^2+y^2-11=-9$, which is not possible. Therefore, $xy \geq 2$, so $\left\lfloor \frac{xy}{2} \right\rfloor \geq 1$. Thus, $k+2-a+b>0$. If $k+2-a+b=1$, then substituting $a=k+1+b$ into $(*)$ gives

$$\begin{aligned}
(k+2)^2 &= (k+1+b)^2+(k+1+b)+b^2+b \\
k^2+4k+4 &= (k^2+2kb+b^2+2k+2b+1)+(k+1+b)+b^2+b \\
0 &= 2b^2+2kb-k+4b-2 \\
0 &= 2b^2+(2b-1)(k+2).
\end{aligned}$$

If $b \geq 1$, then the right side of the equation above is positive, but if $b=0$, then the right side is negative—a contradiction. So $k+2-a+b>1$. . . . . [2 pts]

[4] A perfect square is always 0 or 1 mod 4—an even perfect square $4m^2 \equiv 0 \pmod 4$, whereas an odd perfect square $(2m+1)^2=4m^2+4m+1 \equiv 1 \pmod 4$.

We have now proven that $\left\lfloor \frac{xy}{2} \right\rfloor = (k+2+a-b)(k+2-a+b)$ where both factors are greater than 1. $\left\lfloor \frac{xy}{2} \right\rfloor$ is hence composite.

(Solution by Tian)

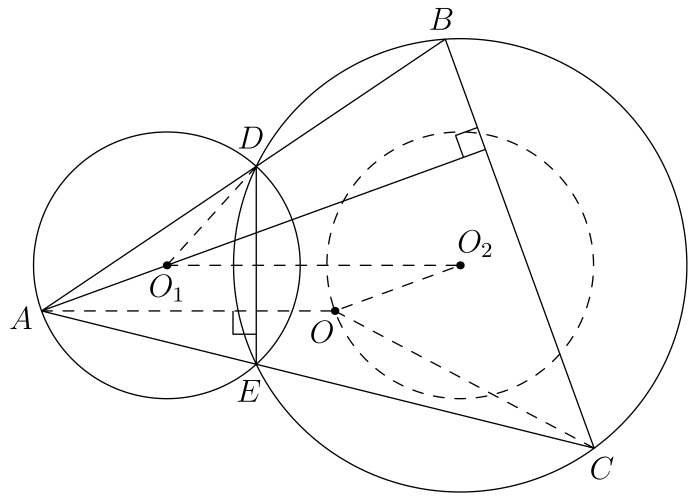
Remark: If $n > 1$, then we also have $\left\lceil \frac{xy}{2} \right\rceil$ is composite. Try proving this yourself—the basic idea is similar to that of proving $\left\lfloor \frac{xy}{2} \right\rfloor$ is composite.

9. [9 pts] Let circles ω_1 and ω_2 intersect at distinct points D and E. Let A distinct from D, E lie on ω_1. Let line AD intersect ω_2 at D and B and let line AE intersect ω_2 at E and C. Prove that as A varies along ω_1 while ω_1 and ω_2 are kept fixed, the circumcenter of $\triangle ABC$ moves along a fixed circle.

   Note: The *circumcenter* of a triangle is the center of the circle that passes through all of its vertices.

(Problem by Zed)

Solution:



WLOG, let the configuration be as shown in the diagram above.

Let the centers of ω_1, ω_2 and $\triangle ABC$'s circumcircle be O_1, O_2 and O, respectively. We claim that the O lies on the circle with center O_2 and radius AO_1. We do so by proving that AO_1O_2O is a parallelogram.

Note that

$$\begin{aligned}
\angle OAC &= 90^\circ - \frac{1}{2}\angle AOC && \text{(since } \triangle AOC \text{ is isosceles)}\\
&= 90^\circ - \angle ABC && \text{(inscribed angle theorem on } \triangle ABC\text{'s circumcircle)}\\
&= 90^\circ - \angle AED, && \text{(property of cyclic quadrilateral } BCED\text{)}
\end{aligned}$$

so $AO \perp DE$. But O_1 and O_2 lie on the perpendicular bisector of DE, meaning $O_1O_2 \perp DE$ and thus

$$AO \parallel O_1O_2. \tag{1}$$

. . . . [3 pts]

In addition,

$$\begin{aligned}
\angle O_1AD &= 90^\circ - \frac{1}{2}\angle AO_1D && \text{(since } \triangle AO_1D \text{ is isosceles)}\\
&= 90^\circ - \angle AED && \text{(inscribed angle theorem on } \omega_1\text{)}\\
&= 90^\circ - \angle ABC, && \text{(property of cyclic quadrilateral } BCED\text{)}
\end{aligned}$$

so $AO_1 \perp BC$. But O and O_2 lie on the perpendicular bisector of BC, meaning $OO_2 \perp BC$ and thus

$$AO_1 \parallel OO_2. \tag{2}$$

.................................[4 pts]

From (1) and (2), we conclude that AO_1O_2O is a parallelogram. Therefore, $OO_2 = AO_1$. As A moves along ω_1 with ω_1, ω_2 kept fixed, the circumcenter O of $\triangle ABC$ thus moves along a fixed circle with center O_2 and radius AO_1. ....[2 pts]

(Solution by Zed)

Remark: After proving that $AO \parallel O_1O_2$, we could've also proven $AO = O_1O_2$ to show that AO_1O_2O is a parallelogram. This can be done through similar triangles: $\triangle ADC \sim \triangle O_1DO_2$ and $\triangle AOC \sim \triangle AO_1D$ due to AA, so $\frac{O_1O_2}{AO} = \frac{O_1O_2}{AC}\frac{AC}{AO} = \frac{O_1D}{AD}\frac{AD}{AO_1} = 1$.

10. [10 pts] Let positive real numbers x, y, z satisfy $x^5y^5 \le 4$ and $xyz = \sqrt{\dfrac{x^5+y^5+z^5}{3}}$. Find the maximum possible value of xyz.

(Problem by Zed)

Solution:

We prove that the answer is $\boxed{2}$.

Let $t = xyz = \sqrt{\frac{x^5+y^5+z^5}{3}} > 0$. Then $x^5+y^5+z^5 = 3t^2$. By the AM-GM inequality,

$$x^5 + y^5 + \frac{z^5}{4} + \frac{z^5}{4} + \frac{z^5}{4} + \frac{z^5}{4} \ge 6\sqrt[6]{(x^5)(y^5)\left(\frac{z^5}{4}\right)\left(\frac{z^5}{4}\right)\left(\frac{z^5}{4}\right)\left(\frac{z^5}{4}\right)}$$

$$x^5+y^5+z^5 \ge 6\sqrt[6]{\frac{x^5y^5z^{20}}{4^4}}$$

.... [3 pts]

$$x^5+y^5+z^5 \ge 6\sqrt[6]{\left(\frac{x^5y^5z^{20}}{4^4}\right)\left(\frac{x^{15}y^{15}}{4^3}\right)} \qquad (\text{since } x^5y^5/4 \le 1)$$

$$x^5+y^5+z^5 \ge 6\sqrt[6]{\frac{x^{20}y^{20}z^{20}}{4^7}} = 6\sqrt[3]{\frac{x^{10}y^{10}z^{10}}{2^7}}$$

$$3t^2 \ge 6\sqrt[3]{\frac{t^{10}}{2^7}}$$

.... [4 pts]

$$t^2 \ge 2\sqrt[3]{\frac{t^{10}}{2^7}}$$

$$t^6 \ge 2^3\left(\frac{t^{10}}{2^7}\right) = \frac{t^{10}}{2^4}$$

$$2^4 \ge t^4$$

$$2 \ge t,$$

i.e., $xyz \le 2$. .... [1.5 pts]

Now, consider $x = 2^{\frac{1}{5}}, y = 2^{\frac{1}{5}}, z = 2^{\frac{3}{5}}$, which satisfies the restriction $xyz = \sqrt{\frac{x^5+y^5+z^5}{3}}$ as $xyz = 2$ and $\sqrt{\frac{x^5+y^5+z^5}{3}} = \sqrt{\frac{2+2+8}{3}} = 2$. In this case, we achieve equality $xyz = 2$.

.... [1.5 pts]

In conclusion, the maximum possible value of xyz is 2.

(Solution by Zed)

Remark: An alternative but equivalent way to use AM-GM is to apply it twice: $x^5 + y^5 + z^5 \ge 2\sqrt{x^5y^5} + z^5 \ge x^{10}y^{10}/4 + z^5/2 + z^5/2 \ge 3\sqrt[3]{x^{10}y^{10}z^{10}/16}$, which gives $3t^2 \ge 3\sqrt[3]{t^{10}/16}$.

11. [11 pts] The country of SJAM has 2021 cities arranged in a circle. Every two adjacent cities are connected by a single highway with a positive integer number of lanes. The Prime Minister of the country, Johnny Mac[5], feels that some highways are too wide and some are too narrow. Therefore, he devised a plan to "even out" the lanes. The plan is to first go to the capital city, and then repeat indefinitely the following operation consisting of two steps:

   - Johnny Mac looks at the two highways connected to the city he's currently in. If the two highways have the same number of lanes, then he does nothing with them. Otherwise, he removes one lane from the wider highway and adds a lane to the other.
   - Johnny Mac goes to the next city in clockwise direction around the circle of cities.

   Define $d(t)$ to be the difference between the widths of the widest and narrowest highways after $t \geq 0$ operations by Johnny Mac. (The "width" of a highway is the number of lanes it has.) Determine the smallest possible value for D satisfying the following property: there always exists a $t_0 \geq 0$ such that $d(t) \leq D$ for all $t \geq t_0$, regardless of the initial widths of the highways.

   (Problem by Zed)

   Solution:

   We claim the answer is $\boxed{2}$. Note that $D \geq 2$, since we can construct an initial configuration of highways for which $d(t) = 2$ for all t:

   $$\ldots, 3, 1, 3, 1, 3, \underline{2}, \underline{1}, 3, 1, 3, 1, \ldots$$

   where there are 1010 highways with 3 lanes, 1010 with 1 lane, and 1 with 2 lanes. (The underlined highways are those connected to the city where Johnny Mac currently resides.) .... [1 pt]

   The above configuration has $d(0) = 3 - 1 = 2$. In addition, after one operation, the configuration becomes

   $$\ldots, 3, 1, 3, 1, 3, 1, \underline{2}, \underline{3}, 1, 3, 1, \ldots$$

   which has $d(1) = 3 - 1 = 2$. After the second operation, the configuration becomes

   $$\ldots, 3, 1, 3, 1, 3, 1, 3, \underline{2}, \underline{1}, 3, 1, \ldots$$

   which is the same as the initial configuration, but shifted to the right by two cities. Therefore, the above cycle of two operations will repeat itself indefinitely, with the configuration at the end of each cycle being a shifted version of the configuration at the start of the cycle. Hence, $d(t) = 3 - 1 = 2$ for all t. .... [0.5 pts]

[5] Mascot of SJAM

Now, we show that $D = 2$ is possible. In other words, we show the existence of $t_0 \geq 0$ for which $d(t) \leq 2$ for all $t \geq t_0$, regardless of the initial widths of the highways.

First, we show that after t_0 operations for some $t_0 \geq 0$, all the remaining operations will occur between highways whose widths differ by at most 1. Consider S, the sum of squares of all highway widths. Note that if $b - a > 1$, then

$$\begin{aligned}(a+1)^2 + (b-1)^2 &= a^2 + 2a + 1 + b^2 - 2b + 1 \\ &= a^2 + b^2 - 2(b - a - 1) \\ &< a^2 + b^2.\end{aligned}$$

Therefore, if Johnny Mac operates on highways with widths a and b ($b > a$) that differ by more than 1, then the sum of squares of the two highway widths will decrease, resulting in a decrease in S. Since S is a positive integer, it can only be decreased finitely many times, meaning that after some point in the future, all remaining operations can only occur between highways whose widths differ by at most 1. .... [5.5 pts]

We now show that the above can only happen if $d(t) \leq 2$ for all $t \geq t_0$. Assume that after the t_0th operation, Johnny Mac is located in a city where the highway behind him (i.e., the one connecting his current city to his previous city) has l lanes. Then the width of the highway in front of him must be l, $l + 1$, or $l - 1$. In each case, the highway in front of him will have l lanes after Johnny Mac's next operation. And as he moves to his next city, this highway becomes the highway behind him, which will have l lanes again. The entire process then repeats itself indefinitely. Thus, at any point in time after Johnny Mac's first t_0 operations, he will always have l lanes behind him and l, $l + 1$, or $l - 1$ lanes in front of him. This means that all highways will have either l, $l + 1$, or $l - 1$ lanes, so $d(t) \leq (l + 1) - (l - 1) = 2$ for all $t \geq t_0$. ..... [4 pts]

In conclusion, the minimum possible value for D is 2.

(Solution by Zed)

12. [12 pts] Consider $\triangle ABC$ with $\angle ABC \neq 90°$ and $\angle ACB \neq 90°$. Let D distinct from B, C lie on line BC. Let E be on line AB such that $BE = DE$. Similarly, let F be on line AC such that $CF = DF$. Prove that if the orthocenter of $\triangle DEF$ and the circumcenter of $\triangle ABC$ do not coincide, then the line joining them is parallel to BC.

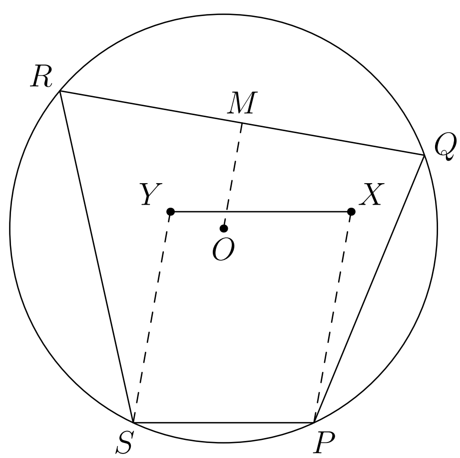
Note: The *orthocenter* of $\triangle DEF$ is the unique point H such that $HD \perp EF$, $HE \perp FD$, and $HF \perp DE$. The *circumcenter* of a triangle is the center of the circle that passes through all of its vertices.

(Problem by Tian)

Solution:

Lemma 1. *In cyclic quadrilateral $PQRS$, X and Y are the orthocenters of $\triangle PQR$ and $\triangle SQR$ respectively. Then $XY \parallel PS$.*

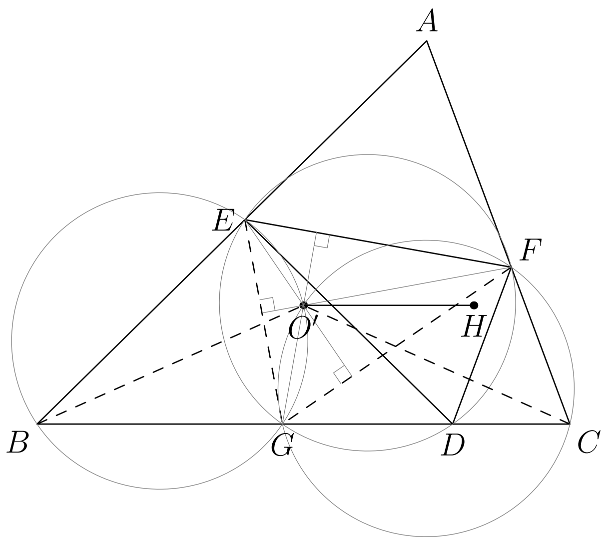
Proof. Let O be the center of the circumcircle of $PQRS$ and let M be the midpoint of QR.



Note that $\overrightarrow{PX} = 2\overrightarrow{OM}$ and $\overrightarrow{SY} = 2\overrightarrow{OM}$ due to this well-known lemma. Therefore, $\overrightarrow{PX} = \overrightarrow{SY}$, so $PXYS$ is a parallelogram and $XY \parallel PS$.

$\square$

.... [3 pts]



WLOG, let the configuration be as shown above.

Let the circumcircle of $\triangle DEF$ intersect line BC at D and G. Let O' be the orthocenter of $\triangle EFG$. We claim O' is the circumcenter of $\triangle ABC$.

Notice that

$$\begin{aligned}
\angle GO'F &= 180^\circ - \angle O'FG - \angle FGO' \\
&= 180^\circ - (90^\circ - \angle FGE) - (90^\circ - \angle EFG) \\
&= \angle FGE + \angle EFG \\
&= 180^\circ - \angle GEF \\
&= \angle FDG && \text{(property of cyclic quadrilateral } DFEG) \\
&= 180^\circ - \angle CDF = 180^\circ - \angle FCG, && (\triangle CDF \text{ is isosceles})
\end{aligned}$$

so C, F, O', G are on the same circle due to this theorem. By symmetry, B, E, O', G are on the same circle. . . . . [4 pts]

Thus, we find

$$\begin{aligned}
\angle O'CB &= \angle O'FG && \text{(inscribed angle theorem on } CFO'G) \\
&= 90^\circ - \angle FGE \\
&= 90^\circ - \angle FDE && \text{(inscribed angle theorem on } DFEG) \\
&= 90^\circ - (180^\circ - \angle EDB - \angle CDF) \\
&= 90^\circ - (180^\circ - \angle DBE - \angle FCD) && (\triangle CDF \text{ and } \triangle BDE \text{ are isosceles}) \\
&= 90^\circ - \angle A.
\end{aligned}$$

Similarly, $\angle CBO' = 90^\circ - \angle A$. This shows that $\angle O'CB = \angle CBO'$, from which $BO' = CO'$. In addition, $\angle BO'C = 180^\circ - \angle O'CB - \angle CBO' = 180^\circ - (90^\circ - \angle A) - (90^\circ - \angle A) = 2\angle A$. By the converse of the inscribed angle theorem, O' is the circumcenter of $\triangle ABC$. . . . . [4 pts]

Now, by applying Lemma 1 with $DFEG$ being $PQRS$, we get $HO' \parallel DG$, and we're done. . . . . [1 pt]

(Solution by Tian)