Second SJAM Mathematics Olympiad

Dec. 21, 2020 – Jan. 17, 2021

1. [1 pt] Given that $x - 3y + 3z = 7$ and $2x + 4y + z = 9$, determine the value of $x + y + z$.

2. [2 pts] Let $ABCD$ be a rectangle. E is on side BC and F is on side DA such that $BEDF$ is a rhombus. If $BE = 2$ and $CE = \sqrt{3}$, determine the size of the acute angle between lines AB and EF.

3. [3 pts] Biscuit is buying 656 biscuits for his boss, Tian. Biscuit needs to get Tian precisely 656 biscuits, no more, no less; otherwise, Biscuit will be fired. Unfortunately, all biscuits in the world come in packs of 7 or 13. How many packs of each kind does Biscuit need to buy to have exactly 656 biscuits for Tian?

4. [4 pts] The equation
$$x^3 - 3kx^2 + (3k^2 + p)x - k^3 - pk = 0,$$
when solved for x, gives three distinct real roots. If two of these roots are a and b $(a \neq b)$, find all possible pairs (p, k). Write your answers in terms of a and b.

5. [5 pts] Show that 8 is the minimum number of cards Cookie needs to draw from a standard 52-card deck to guarantee that he has two cards where the value on one of them is a multiple of the value on the other card. (Assume that A = 1, J = 11, Q = 12, and K = 13.)

6. [6 pts] Every point of the plane is colored one of two colors: gamboge and razzmatazz. Prove that, for all $d > 0$, there exists a monochromatic isosceles triangle with positive area and leg length d.

   Note: A *monochromatic* triangle is one whose vertices are of the same color.

7. [7 pts] Find the maximum possible value for positive integer n such that the following property holds: there exists some positive integer M such that $p^2 - 1$ is divisible by n for all prime numbers $p \geq M$.

8. [8 pts] If $n^2 + 6n + 11 = x^2 + y^2$ for positive integers n, x and y, prove that $\left\lfloor \frac{xy}{2} \right\rfloor$ is composite.

   Note: $\lfloor t \rfloor$ represents the greatest integer less than or equal to t.

9. [9 pts] Let circles ω_1 and ω_2 intersect at distinct points D and E. Let A distinct from D, E lie on ω_1. Let line AD intersect ω_2 at D and B and let line AE intersect ω_2 at E and C. Prove that as A varies along ω_1 while ω_1 and ω_2 are kept fixed, the circumcenter of $\triangle ABC$ moves along a fixed circle.

   Note: The *circumcenter* of a triangle is the center of the circle that passes through all of its vertices.

10. [10 pts] Let positive real numbers x, y, z satisfy $x^5y^5 \leq 4$ and $xyz = \sqrt{\dfrac{x^5+y^5+z^5}{3}}$. Find the maximum possible value of xyz.

11. [11 pts] The country of SJAM has 2021 cities arranged in a circle. Every two adjacent cities are connected by a single highway with a positive integer number of lanes. The Prime Minister of the country, Johnny Mac[1], feels that some highways are too wide and some are too narrow. Therefore, he devised a plan to "even out" the lanes. The plan is to first go to the capital city, and then repeat indefinitely the following operation consisting of two steps:

    - Johnny Mac looks at the two highways connected to the city he's currently in. If the two highways have the same number of lanes, then he does nothing with them. Otherwise, he removes one lane from the wider highway and adds a lane to the other.
    - Johnny Mac goes to the next city in clockwise direction around the circle of cities.

    Define $d(t)$ to be the difference between the widths of the widest and narrowest highways after $t \geq 0$ operations by Johnny Mac. (The "width" of a highway is the number of lanes it has.) Determine the smallest possible value for D satisfying the following property: there always exists a $t_0 \geq 0$ such that $d(t) \leq D$ for all $t \geq t_0$, regardless of the initial widths of the highways.

12. [12 pts] Consider $\triangle ABC$ with $\angle ABC \neq 90°$ and $\angle ACB \neq 90°$. Let D distinct from B, C lie on line BC. Let E be on line AB such that $BE = DE$. Similarly, let F be on line AC such that $CF = DF$. Prove that if the orthocenter of $\triangle DEF$ and the circumcenter of $\triangle ABC$ do not coincide, then the line joining them is parallel to BC.

    Note: The *orthocenter* of $\triangle DEF$ is the unique point H such that $HD \perp EF$, $HE \perp FD$, and $HF \perp DE$. The *circumcenter* of a triangle is the center of the circle that passes through all of its vertices.

All problems by the SJAM Mathletes Problem Committee (Zed[2], Tian[3], Biscuit[4])

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