J/STOW #6: Invariant Principle

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1 Basic principles

1. Find a property or value that is invariant (doesn't change).
2. Find a value that changes in the same way (e.g. always decreases or increases).

2 Practice problems

1. Zed bought 10 candies for his two children, Eks and Why, who don't have any candies at the start. Zed is going to share his candies with his children, but in a stupid way. One of Zed and Eks gives the other a candy. One of Eks and Why then gives the other a candy. One of Why and Zed then gives the other a candy. Then the entire process above cycles over and over again indefinitely. Is it possible that at some point,

   i) Zed, Eks and Why each have the same number of candies?

   ii) Zed has 4 candies, Eks has 3 candies, and Why has 3 candies?

2. Jimmy is bored. He writes down $5, 8, 14$ on the blackboard. Each time Jimmy gets the numbers a, b, c on the blackboard, he erases them and replaces them with $\frac{b+c}{2}, \frac{c+a}{2}, \frac{a+b}{2}$. Can Jimmy ever at some point get three 8's on the blackboard?

3. Jimmy is bored. He writes down $5, 8, 14$ on the blackboard. Each time Jimmy gets the numbers a, b, c on the blackboard, he erases them and replaces them with $\frac{b+c}{2}, \frac{c+a}{2}, \frac{a+b}{2}$. Can Jimmy ever at some point get three 9's on the blackboard? *(SJAMMO 2019 J2)*

4. There are a bunch of points in the plane, no three of which are collinear. Those points are partitioned into pairs, and the two points of each pair are connected with a line segment. On each move, CB can take take two intersecting line segments and "uncross" them: if segments AB and CD intersect, he erases them and draws the segments AC and BD. Prove that eventually all line segments intersect no other line segment. *(Classical)*

5. Tian starts with three numbers $3, 4, 5$ on the blackboard. During each move, Tian can choose two numbers on the blackboard and replace them with their sum and non-negative difference, and then multiples the other number by $\sqrt{2}$. Is there a sequence of moves that will result in three numbers that are all between 47 and 65, inclusive?

6. There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines.

   At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey? *(EGMO 2017 P3)*

Hints. . .

1. i) Total number of candies. ii) Parity (even/odd).
2. Add.
3. Subtract.
4. Sum of all lengths.
5. Square.
6. Alternating coloring of regions.