Answers and Brief Explanations to JSTOW #5: Principle of Inclusion and Exclusion

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1 Practice Problems And Solutions

1. During exam season, 80% of students passed English and 85% passed Math, while 75% passed both English and Mathematics. If 45 students failed both exams, find the total number of students.

   Let the set of students who passed the English exam be A, let the set of students who passed the Math exam be B, and let n be the total number of students. Thus $|A| = 0.8n$, $|B| = 0.85n$, and $|A \cap B| = 0.75n$. We know $|A \cup B| = |A| + |B| - |A \cap B| = 0.8n + 0.85n - 0.75n = 0.9n$. Thus the 90% of the students passed at least one exam, which means 10% didn't pass any of the exams. We also know that 45 people didn't pass any of the exams, which means there must have been 450 people in total.

2. In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle?

   By PIE, the number of adults who own both cars and motorcycles is $331+45-351 = 25$. Out of the 331 car owners, 25 of them own motorcycles and $331 - 2 = 306$ of them don't.

3. Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

   Consider the three-digit arrangement, $\overline{aba}$. There are 10 choices for a and 10 choices for b (since it is possible for $a = b$), and so the probability of picking the palindrome is $\frac{10\times10}{10^3} = \frac{1}{10}$. Similarly, there is a $\frac{1}{26}$ probability of picking the three-letter palindrome.

   By the Principle of Inclusion-Exclusion, the total probability is

$\frac{1}{26} + \frac{1}{10} - \frac{1}{260} = \frac{35}{260} = \frac{7}{52} \quad \Longrightarrow \quad 7 + 52 = 59$

4. There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

   By PIE, we have $|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|$. The number of people in at least two sets is $\sum |A_i \cap A_j| - 2|A_1 \cap A_2 \cap A_3| = 9$. So, $20 = (10 + 13 + 9) - (9 + 2x) + x$, which gives $x = 3$.

5. In a room, 2/5 of the people are wearing gloves, and 3/4 of the people are wearing hats. What is the minimum number of people in the room wearing both a hat and a glove?

   Let x be the number of people wearing both a hat and a glove. Since the number of people wearing a hat or a glove must be whole numbers, the number of people in the room must be a multiple of $\mathrm{lcm}(4, 5) = 20$. Since we are trying to find the minimum x, we must use the smallest possible value for the number of people in the room. Similarly, we can assume that there are no people present who are wearing neither of the two items since this would unnecessarily increase the number of people in the room. Thus, we can say that there are 20 people in the room, all of which are wearing at least a hat or a glove.

   It follows that there are $\frac{2}{5} \cdot 20 = 8$ people wearing gloves and $\frac{3}{4} \cdot 20 = 15$ people wearing hats. Then by applying the Principle of Inclusion Exclusion (PIE), the total number of people in the room wearing either a hat or a glove or both is $8 + 15 - x = 23 - x$. Since we know that this equals 20, it follows that $23 - x = 20$, which implies that $x = 3$. Thus, 3 is the correct answer.