J/STOW #5: Principle of Inclusion and Exclusion

Cristian Bicheru   Tian Chen

February 14, 2020

All of the fancy terms in this paper will have links to an appropriate definition. The links will also be on the Mathletes Google Classroom if you are too lazy to type them out.

1 Basic principles

1. **Motivation:** What is PIE? Why do we need it anyways?? The Principle of Inclusion and Exclusion (or sometimes Inclusion–Exclusion Principle) is a counting technique which makes it easy for us to compute the cardinality of (or number of elements in) the union between two or more sets.

2. **Example Problem:** Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. **Example Problem Solution:** If we think about that problem in terms of sets, the answer is simply

$$P = \frac{|\{\text{set of licence plates containing at least one palindrome}\}|}{|\{\text{set of all licence plates}\}|}$$

Let A be the set of all licence plates where the letters are palindromes and let B be the set of all licence plates where the numbers are palindromes. Thus:

$$P = \frac{|A \cup B|}{|\{\text{set of all licence plates}\}|}$$

Which, using PIE, simplifies to:

$$P = \frac{|A| + |B| - |A \cap B|}{|\{\text{set of all licence plates}\}|}$$

Which is now possible to evaluate.

4. **Introduction:** So I mentioned that $|A \cup B| = |A| + |B| - |A \cap B|$. But why is this true? Why isn't it just $|A \cup B| = |A| + |B|$? Well if you think about it in terms of Venn-diagrams, if you just add the cardinalities of sets A and B, you actually double count the intersection of the two, since the intersection is a part of set A and set B and you add the cardinality of each one. But that's an easy fix! We simply subtract the cardinality of the intersection once, and now it's only been counted once. This also holds with more than two sets, here's an example with three sets A, B, and C. Here we can see that if we simply do $|A| + |B| + |C|$, we double

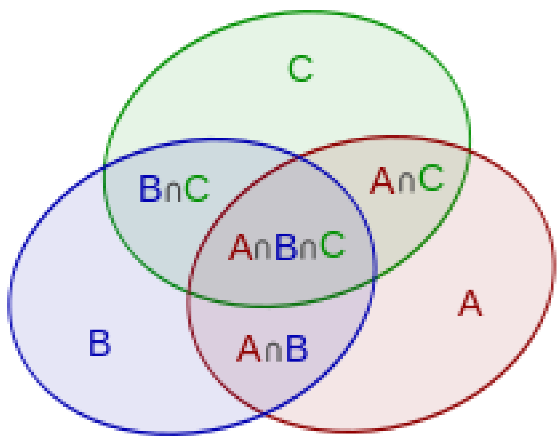

Figure 1: Union of Three Sets.

count $|B \cap C|$, $|A \cap B|$, and $|A \cap C|$, and triple count $|A \cap B \cap C|$. Okay, so then $|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| - 2 \cdot |A \cap B \cap C|$ right? Actually, no! Because $|B \cap C|$, $|A \cap B|$, and $|A \cap C|$ all contain $|A \cap B \cap C|$. So when we subtract each of those intersections, we actually completely remove $|A \cap B \cap C|$ from the result. Thus instead of subtracting it twice, we have to add it back once to make up for that. Therefore $|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$, which is correct!

5. **General Form:** Most of the time, the formula for two and three sets is enough. However, in some really abstract problems, you might need higher order versions of the principle. Thus I present to you the general form of PIE for finite sets $A_1, A_2, ..., A_n$:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots \\ + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

Honestly, it's kind of hard to make sense of that, but if you understood what we did earlier to obtain the two formulae, you can probably derive higher order formulae without memorizing the general form.

2 Practice Problems

1. During exam season, 80% of students passed English and 85% passed Math, while 75% passed both English and Mathematics. If 45 students failed both exams, find the total number of students.

2. In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle?

3. There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

4. Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. In a room, 2/5 of the people are wearing gloves, and 3/4 of the people are wearing hats. What is the minimum number of people in the room wearing both a hat and a glove?