STOW #3 Functional Equations Solutions

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NOTICE: For question 11, it should have said $f(n+1) > f(n)$.

1. (Traditional) A function f is called an *involution* if $f(f(x)) = x$. Show that f is bijective.

   If $f(a) = f(b)$, then $f(f(a)) = f(f(b))$ so $a = b$. This proves f is injective. For any x, let $y = f(x)$. Then $f(y) = x$ which proves f is surjective. Hence, f is bijective.

2. (Traditional) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all real x, y.

   When $x = y$, $f(x^2) = f(x)^2 \geq 0$. This tells us that when the input of f is ≥ 0, the output of f is ≥ 0. Thus, $f(x) = cx$ for some constant c since f satisfies Cauchy. We find that $c = c^2$ so $c = 0, 1$. Thus, $f(x) = 0$ and $f(x) = x$ are the only solutions.

3. (SJAMMO 2019) Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $2f(n) = n + f(f(n))$ and $f(0) = 1$ for all $n \in \mathbb{N}_0$.

   Suppose $f(n) = n + 1$ for some $n \in \mathbb{N}_0$. Then,

   $$\begin{aligned} 2f(n) &= n + f(f(n)) \\ \iff 2(n+1) &= n + f(n+1) \\ \iff n + 2 &= f(n+1) \end{aligned}$$

   The base case is $f(0) = 1$. Thus, $f(n) = n + 1$ for all $n \in \mathbb{N}_0$.

4. (Somewhere on the internet[1]) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x) + y) = x + f(f(y))$ for all real x, y.

   If $f(a) = f(b)$, then $a + f(f(y)) = f(f(a) + y) = f(f(b) + y) = b + f(f(y))$ so f is injective. Setting $x = 0$ gives $f(f(0) + y) = f(f(y)) \implies f(0) + y = f(y)$. In other words, $f(x) = x + c$ for some real constant c.

5. (Jensen's Functional Equation) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $\frac{f(x)+f(y)}{2} = f\left(\frac{x+y}{2}\right)$ for all real x, y.

[1] You can't just search these things up you know?

Let $g(x) = f(x) - f(0)$. Then we know $g(0) = 0$. We find that $\frac{g(x)+g(y)}{2} = g(\frac{x+y}{2})$ holds. Setting $y = 0$ gives $\frac{g(x)}{2} = g(\frac{x}{2}) \implies g(x) = 2g(\frac{x}{2})$. This means $g(x) + g(y) = 2g(\frac{x+y}{2}) = g(x+y) \implies g(x) = cx$ for some constant c since g is continuous. Therefore, $f(x) = cx + d$ for some constants c and d.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + y) = f(x)^2 + f(y)$ for all real x, y.

   When $x = y = 0$, then $f(0) = f(0)^2 + f(0) \implies f(0) = 0$. Setting $y = 0$ gives $f(x^2) = f(x)^2$. Using this on the original equation gives $f(x^2 + y) = f(x^2) + f(y)$. Since f is bounded (See problem 2), we may deduce that $f(x) = cx$ for some constant c and using this gives $c = 1$. (To see why the Cauchy argument works with a non-negative real, see problem 10.) This gives us the only solution of $f(x) = x$.

7. (South Africa 1997) Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + f(y)) = y + f(x)$ for all $x, y \in \mathbb{Q}$.

   If $f(a) = f(b)$, then $a + f(x) = f(x + f(a)) = f(x + f(b)) = b + f(x)$ so f is injective. Setting $y = 0$, we get $f(x + f(0)) = f(x) \implies f(0) = 0$. Setting $x = 0$ gives $f(f(y)) = y + f(0) = y$. Setting $y = f(t)$ (it turns out that y can be any rational) gives $f(x + t) = f(t) + f(x) \implies f(x) = cx$ for some rational constant c. Plugging this into the original equation yields $c(x + cy) = y + cx \implies c = \pm 1$ (when $y \neq 0$ but $f(0) = 0$ anyways). Thus, $f(x) = x$ and $f(x) = -x$ are the only solutions.

   Note: The original problem is over $\mathbb{Z}$.

8. (Traditional) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

   (a) $f(xy) = f(x)f(y)$

   First, we find $f(0)$. Clearly, $f(0) = f(0)^2 \implies f(0) = 0$ or $f(0) = 1$. If $f(0) = 1$, then $f(x)f(0) = f(0x) \implies f(x) = 1$ so assume $f(0) = 0$.

   We find that $f(x) = f(1)f(x)$ so either $f(x) = 0$ or $f(1) = 1$. Assume that f is not identically 0. Then, $f(-1)^2 = f(1) = 1$ so $f(-1) = 1$ or $f(-1) = -1$.

   If $f(-1) = 1$, then $f(x)f(-1) = f(-x) \implies f(x) = f(-x)$ so f is even.

   If $f(-1) = -1$, then $f(x)f(-1) = f(-x) \implies -f(x) = f(-x)$ so f is odd.

   Now, we only consider $f : \mathbb{R} \to \mathbb{R}^+$.

   Let $g(x) = \ln(f(e^x))$ (since ln is only defined for positive reals). Then $g(x + y) = \ln(f(e^{x+y})) = \ln(f(e^x)f(e^y)) = \ln(f(e^x)) + \ln(f(e^y)) = g(x) + g(y)$ so g satisfies Cauchy. Since g is continuous, $g(x) = cx$ for some constant c so $f(e^x) = e^{g(x)} = e^{cx} = (e^x)^c$. Thus, $f(x) = f(e^{\ln x}) = (e^{\ln x})^c = x^c$.

   We see that $f(x)$ is always positive when x is positive so what if x is negative? If f is even, we find $f(x) = |x|^c$. If f if odd, then $f(x) = \text{sgn}(x) \cdot |x|^c$.

   Thus, we find $f(x) = 1, f(x) = 0, f(x) = |x|^c$, and $f(x) = \text{sgn}(x) \cdot |x|^c$ where c is some positive real constant. (Why can't c be negative or zero?)

(b) $f(x+y) = f(x)f(y)$

We immediately get $f(0) = f(0)^2 \implies f(0) = 0$ or 1. If $f(0) = 0$, setting $y = 0$ gives $f(x) = 0$ so assume otherwise.

Setting $y = -x$ gives, $f(0) = f(x)f(-x) \implies f(-x) = \frac{1}{f(x)}$.

Now we look at f over $\mathbb{R}^+$.

Let $g(x) = \ln(f(x))$. Then $g(x+y) = \ln(f(x+y)) = \ln(f(x)f(y)) = \ln(f(x)) + \ln(f(y)) = g(x) + g(y)$ so g satisfies Cauchy. Since g is continuous, $g(x) = cx$ for some constant c. Therefore, $f(x) = e^{g(x)} = e^{cx}$.

We see that $f(-x) = \frac{1}{f(x)} = (e^{cx})^{-1} = e^{-cx}$ so $f(x) = e^{cx}$ for all reals.

Thus, the only solutions are $f(x) = 0$ and $f(x) = e^{cx}$ for any real constant c.

(c) $f(xy) = f(x) + f(y)$

Clearly, $f(0) = 2f(0) \implies f(0) = 0$.

Let $g(x) = f(e^x)$. Then, $g(x+y) = f(e^{x+y}) = f(e^x e^y) = f(e^x) + f(e^y) = g(x) + g(y)$ so g satisfies Cauchy. Since g is continuous, $g(x) = cx$ for some constant c so $f(x) = f(e^{\ln x}) = g(\ln x) = c \ln x$ for $x > 0$.

Since $\ln x$ approaches $-\infty$ as x approaches 0, there are no solutions for f other than $f(x) = 0$.

Suppose f is from the non-zero reals to the reals. Note that $f(1) = 0$ and so $f(1) = f(-1) + f(-1) \implies f(-1) = 0$. For $x > 0$, $f(-x) = f(-1) + f(x) = f(x)$ so f is even. Thus, $f(x) = c \cdot \ln|x|$ for any real constant c is the only solution.

9. (Singapore 1999) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $bf(a) - af(b) = ab(a^2 - b^2)$ for all real a, b.

   When $a = 0$, $bf(0) = 0$ so $f(0) = 0$.

   Now, we assume $a, b \neq 0$ and rearrange the original equation into $\frac{f(a)}{a} - a^2 = \frac{f(b)}{b} - b^2$. Since each side is independent on the other, we conclude that $\frac{f(x)}{x} - x^2 = c \implies f(x) = x^3 + cx$ for some constant c.

   Note: The original problem is $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$. Setting $a = x + y$ and $b = x - y$ gives our problem so be clever about your substitutions!

10. (Traditional) Prove Cauchy's function equation over the integers. Then over the rationals.

    Cauchy's functional equation is $f(x+y) = f(x) + f(y)$.

    We see that clearly $f(0) = 0$.

    For the integers, we use Cauchy repeatedly on $f(an) = f(a + a + \cdots + a)$ where $n \in \mathbb{Z}^+$. This gives $f(an) = nf(a)$. Setting $a = 1$ gives $f(n) = cn$ where $c = f(1)$. Cauchy gives $f(n) + f(-n) = 0$ so $f(-n) = -f(n) = -cn$. Thus, $f(n) = n$ for all $n \in \mathbb{Z}$.

For the rationals, replacing a with $\frac{a}{n}$ in $f(an) = nf(a)$ gives $f(a) = nf(\frac{a}{n})$. Multiplying by $\frac{m}{n}$ where $m \in \mathbb{Z}^+$ gives $\frac{m}{n}f(a) = mf(\frac{a}{n}) = f(a\frac{m}{n})$. Setting $a = 1$ gives $f(\frac{m}{n}) = c\frac{m}{n}$ where $c = f(1)$. With a similar argument for the integers, we see that this is true for all $m \in \mathbb{Z}$. Thus, $f(q) = cq$ for all $q \in \mathbb{Q}$.

11. (Canada 1969) Find all functions $f : \mathbb{N}^+ \to \mathbb{N}^+$ such that for all $m, n \in \mathbb{N}^+$: $f(2) = 2$, $f(n+1) > f(n)$ and $f(mn) = f(m)f(n)$.

    We see that $f(2n) = f(2)f(n) = 2f(n)$ for all naturals n. This implies that for all naturals k, $f(2^k) = 2^k$.

    Look at the numbers $f(2^k + 1), f(2^k + 2), \cdots, f(2^{k+1} - 1)$. There are $2^k - 1$ of them and they all lie between $f(2^k) = 2^k$ and $f(2^{k+1}) = 2^{k+1}$.

    But there are only $2^k - 1$ naturals between 2^k and 2^{k+1}. Since $f(2^k + 1) < f(2^k + 2) < \cdots < f(2^{k+1} - 1)$, we conclude that $f(n) = n$ for all $n \geq 2$.

    We get $f(1) = 1$ from $f(1) = f(1)^2$ so $f(n) = n$ is the only solution.

    Note: The original problem has $f(m) > f(n)$ when $m > n$ instead of $f(n+1) > f(n)$. It's easy to show one using the other.

12. (Japan 2004) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(x) + f(y)) = f(x)^2 + y$ and $f(0) = 0$ for all real x, y.

    Setting $y = 0$ gets $f(xf(x)) = f(x)^2$. Setting $x = 0$ gives $f(f(y)) = y$. Setting $x = f(x)$ and $y = 0$ gives $f(xf(x)) = x^2 \implies f(x)^2 = x^2 \implies f(x) = \pm x$ for any particular value of x.

    Suppose there exist non-zero a, b such that $f(a) = a$ and $f(b) = -b$. Then, setting $x = a$ and $y = b$ gives $f(a^2 - b) = a^2 + b$ but $f(a^2 - b) = \pm(a^2 - b)$.

    If $a^2 + b = a^2 - b$, then $b = 0$. If $a^2 + b = -a^2 + b$, then $a = 0$. Both are a contradiction so no such a, b exist.

    We may conclude that $f(x) = x$ and $f(x) = -x$ are the only solutions.

    Bonus: show $f(0) = 0$ without assuming it.

    Setting $x = 0$ gives $f(f(y)) = f(0)^2 + y$. Since the RHS can be any real, f is surjective.

    We see that x is wrapped in $f(x)$ so if $f(x) = 0$, all the x's in the equation disappear. Since f is surjective, let $f(z) = 0$. Setting $x = z$, we get $f(f(y)) = y$. Setting $y = z$ in the previous equation gives $f(0) = z$. Setting $x = y = 0$ gives $f(z) = z^2 \implies z = 0$.

    Note: Even though finding $f(0) = 0$ is not necessary to solve this problem, having goals like this can help guide you to a solution.