STOW #3: Functional Equations

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October 18, 2019

1 Review of Number Sets

The following are the international standards for denoting number sets:

- $\mathbb{N} = \{ 0, 1, 2, \ldots \}$ or $\{ 1, 2, \ldots \}$ is the set of naturals (but this is ambiguous to whether it includes zero or not).
- $\mathbb{N}_0 = \{ 0, 1, 2, \ldots \}$ is the set of naturals including zero.
- $\mathbb{N}^+ = \{ 1, 2, 3, \ldots \}$ is the set of non-zero naturals.
- $\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$ is the set of integers. $\mathbb{Z}^+$ is the set of positive integers. *Note: Please don't use $\mathbb{I}$ to denote the integers since it is not the international standard. See link.*
- $\mathbb{Q}$ is the set of rationals: $\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m, n \in \mathbb{Z}, n \neq 0 \right\}$.
- $\mathbb{R}$ is the set of reals. $\mathbb{R}^+$ is the set of positive reals.

2 Functions

A *function* takes a set of inputs and returns some unique output for each input.

Definition. The *domain* of a function is the set of inputs for which the function is defined. The *codomain* of a function is a set that contain all outputs of the function, but it might also contain other elements.

If X is the domain of f and Y is the codomain of f, then this is expressed as $f : X \to Y$.

Definition. The *range* of a function is the set of all possible outputs of the function. In other words, the range of function $f : X \to Y$ is $R = \{ f(x) \mid x \in X \}$.

From the definition of the codomain, we have that $R \subseteq Y$, which means that the range is always a subset of the codomain.

For example, take the function $f(x) = x^2$. A domain could be $\mathbb{R}$ and a codomain could also be $\mathbb{R}$. Notice that the range of f, which is $\mathbb{R}_{\geq 0}$ (all non-negative reals), is different from (but a subset of) the codomain.

Note: You might be more familiar with the a domain written in the form $D = \{\, x \mid \text{blahblah} \,\}$ and the range written in the form $R = \{\, y \mid \text{blahblah} \,\}$. However, learn to be comfortable with domains and ranges not written in that form.

Definition. A function is *injective* means $f(a) = f(b)$ implies $a = b$. A function is *surjective* if for every y in the codomain of f, there exists x such that $f(x) = y$ (in other words, the codomain *is* the range). A function is *bijective* if it is both injective and surjective.

Definition. A function f is monotonically increasing (or non-decreasing) if for $x \geq y$, $f(x) \geq f(y)$. Similarly, f is monotonically decreasing (or non-increasing) if for $x \geq y$, $f(x) \leq f(y)$.

A functional equation is an equation where you solve for a function.

3 Cauchy's Functional Equation

Cauchy's functional equation is $f(x + y) = f(x) + f(y)$.

All solutions satisfy $f(x) = cx$ for some constant c (i.e. f is linear) if

- $f : \alpha\mathbb{Q} \to \mathbb{R}$ where $\alpha\mathbb{Q} = \{\, \alpha q \mid q \in \mathbb{Q} \,\}$ and $\alpha \in \mathbb{R}$;
- $f : \mathbb{Q} \to \mathbb{Q}$ (replacing $\mathbb{Q}$ with $\mathbb{Z}$ and $\mathbb{N}$ would yield the same solutions except the restrictions on c are different).
- $f : \mathbb{R} \to \mathbb{R}$ and any one of the following is true:
  - f is continuous on any interval (even only at a point).
  - f is monotonic on any interval.
  - f is bounded on any interval, meaning that, for all x in a finite interval or above/below a constant, all the values of $f(x)$ fall in some finite interval or are above/below a constant.

    These properties are important because there exist *pathological* solutions but they are beyond the scope of this handout.

4 The Pointwise Trap

If you find that for a function f, $(f(x) - g(x))(f(x) - h(x)) = 0$ for all x, THIS DOES NOT MEAN $f(x) = g(x)$ FOR ALL x OR $f(x) = h(x)$ FOR ALL x. That is only true for a SPECIFIC value of x, NOT ALL values. (It is indeed true that for each value of x, $f(x) = g(x)$ or $f(x) = h(x)$.)

5 Problems

In very rough order of difficulty:

1. A function f is called an *involution* if $f(f(x)) = x$. Show that f is bijective.
2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all real x, y.
3. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $2f(n) = n + f(f(n)))$ and $f(0) = 1$ for all $n \in \mathbb{N}_0$.
4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x) + y) = x + f(f(y))$ for all real x, y.
5. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $\frac{f(x)+f(y)}{2} = f\left(\frac{x+y}{2}\right)$ for all real x, y.
6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + y) = f(x)^2 + f(y)$ for all real x, y.
7. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + f(y)) = y + f(x)$ for all $x, y \in \mathbb{Q}$.
8. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,
   (a) $f(xy) = f(x)f(y)$
   (b) $f(x + y) = f(x)f(y)$
   (c) $f(xy) = f(x) + f(y)$
9. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $bf(a) - af(b) = ab(a^2 - b^2)$ for all real a, b.
10. Prove Cauchy's function equation over the integers. Then over the rationals.
11. Find all functions $f : \mathbb{N}^+ \to \mathbb{N}^+$ such that for all $m, n \in \mathbb{N}^+$: $f(2) = 2$, $f(n + 1) > n$ and $f(mn) = f(m)f(n)$.
12. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(x) + f(y)) = f(x)^2 + y$ and $f(0) = 0$ for all real x, y. Bonus: show $f(0) = 0$ without assuming it.

(Sources will be cited in the solutions handout.)

6 Hints

1. It's not *that* hard.
2. $f(x^2) = f(x)^2 \geq 0$ when $x = y$. What does this tell you about f?
3. Find $f(1), f(2), f(3), \ldots$ Prove this pattern with induction.
4. When $x = 0$ we get $f(f(0) + y) = f(f(y))$. Wouldn't it be great if f was injective?
5. $f(0)$ need not be 0. Why don't we force it to be? Let $g(x) = f(x) - f(0)$. Notice how the original equation will still hold if f is replaced with g.
6. Show that $f(x^2) = f(x)^2$.
7. When $y = 0$, $f(x + f(0)) = f(x)$. Wouldn't it be great if f was injective?
8. That STOW on logarithms might be helpful *cough* *cough*.
9. Try to get a symmetrical expression only dependent on a on one side and b on the other side.
10. Use Cauchy repeatedly on $f(an) = f(a + a + \cdots + a)$ where $n \in \mathbb{Z}^+$.
11. Prove that if $f(a) = a$, $f(b) = b$ and $a \geq b$, then for all $a \leq k \leq b$, $f(k) = k$.
12. Show that $f(x)^2 = x^2$.