JTOW #3: Basic Inequalities

Zed Li

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1 Basic principles

Note 1: All variables represent real numbers unless otherwise noted.

Note 2: Also, for the sake of simplicity, I only wrote $<$ or $>$ for properties 5 to 12 instead of $\leq$ or $\geq$. This is because the case of equality should be straightforward.

1. $a < b$ is equivalent to $b > a$; $a \leq b$ is equivalent to $b \geq a$.
2. Any of the following implies that $a < c$: 1) $a < b < c$; 2) $a \leq b < c$; 3) $a < b \leq c$.
3. $a \leq b \leq c$ implies that $a \leq c$; $a \leq b \leq a$ implies that $a = b$.
4. **Addition rule:** $a < b$ implies that $a + c < b + c$. Analogously, $a \leq b$ implies that $a + c \leq b + c$.
5. **Multiplication rule:** It is given that $a < b$. If $c > 0$, then $ac < bc$; if $c < 0$, then $ac > bc$ *($\leftarrow$ IMPORTANT! It is very easy to forget to change the direction of the inequality sign ($<$ to $>$ or vice versa) when both sides are multiplied by a negative number)*. An analogous statement can be made where $<$ and $>$ are changed to $\leq$ and $\geq$, respectively.
6. **Division rule:** It is given that $a < b$.
   (a) If $c > 0$:
   if $0 < a < b$, then $\frac{c}{a} > \frac{c}{b} > 0$;
   if $a < 0 < b$, then $\frac{c}{a} < 0 < \frac{c}{b}$;
   if $a < b < 0$, then $0 > \frac{c}{a} > \frac{c}{b}$.
   (b) If $c < 0$:
   if $0 < a < b$, then $\frac{c}{a} < \frac{c}{b} < 0$.
   if $a < 0 < b$, then $\frac{c}{a} > 0 > \frac{c}{b}$;
   if $a < b < 0$, then $0 < \frac{c}{a} < \frac{c}{b}$.
   Trick to remembering these: Don't remember these. Draw the graph of $y = \frac{c}{x}$. Remember what the graph looks like (it looks like this for $c > 0$ and this for $c < 0$). Then, compare the y coordinates at $x = a$ and $x = b$ for $a < b$.

7. **Power rule:** If $0 < a < b$, then $0 < a^c < b^c$ for $c > 0$ while $0 < b^c < a^c$ for $c < 0$. However, if one of a or b is less than 0, then we have to consider 4 cases:

   (a) If c is a positive even integer, then $|a| < |b|$ implies that $a^c < b^c$.

   (b) If c is a negative even integer, then $|a| < |b|$ implies that $a^c > b^c$.

   (c) If c is a positive odd integer, then $a < b$ implies that $a^c < b^c$.

   (d) If c is a negative odd integer, then:
   if $0 < a < b$, then $\frac{c}{a} < \frac{c}{b} < 0$.
   if $a < 0 < b$, then $\frac{c}{a} > 0 > \frac{c}{b}$;
   if $a < b < 0$, then $0 < \frac{c}{a} < \frac{c}{b}$.
   Trick to remembering these: Don't remember these. Again, draw the graph of $y = x^c$ in, for example, Desmos (click on "add slider" to get a slider with which you can change the value of c). Remember what the graphs look like for different values of c. As before, compare the y coordinates at $x = a$ and $x = b$ for $a < b$.

8. **Exponent rule:** It is given that $a < b$. If $0 < c < 1$, then $c^a > c^b$; if $c > 1$, then $c^a < c^b$.

   Trick to remembering these: Again, don't remember these. Draw the graph of $y = c^x$, and remember what the graphs look like for $0 < c < 1$ and $c > 1$. Then, compare the y coordinates at $x = a$ and $x = b$ for $a < b$.

9. $|a| < b$ if and only if $-b < a < b$; $|a| > b \geq 0$ if and only if $a < -b$ or $a > b$; $|a| > b < 0$ is always true for any real a.

10. $|a| < |b|$ if and only if $a^2 < b^2$.

11. $\dfrac{x_1x_2\cdots x_m}{y_1y_2\cdots y_n} > 0$ means that an even number of $x_1, x_2, \ldots, x_m, y_1y_2\cdots y_n$ are negative, while $\dfrac{x_1x_2\cdots x_m}{y_1y_2\cdots y_n} < 0$ means that an odd number of $x_1, x_2, \ldots, x_m, y_1y_2\cdots y_n$ are negative.

12. $x > 0$ if and only if $\frac{1}{x} > 0$; $x < 0$ if and only if $\frac{1}{x} < 0$.

13. **Common methods include but are not restricted to:**

    (a) *Factoring:* After factoring, you can use property 11. See TOW #1.

    (b) *Considering different cases:* As seen through properties 6, 7 and 8, inequality properties involve a lot of cases. Therefore, it is often necessary to consider several different cases when solving or proving an inequality.

    (c) *Getting rid of absolute values:* There are several ways to do this. Some examples include:

       i. using properties 9 and 10;

       ii. considering different cases (i.e. $|x| = x$ for $x \geq 0$ while $|x| = -x$ for $x < 0$).

2 Practice problems

If you're stuck on a problem, you can scroll to the next page to get some hints.

Note: If not mentioned, a variable is assumed to be representing a real number.

1. Solve for x: $x^n > 1$ for integer n.
2. Solve for x: $x^2 + 2x + 2 \geq 0$.
3. Solve for x: $x^2 - 3|x| + 2 > 0$.
4. Solve for x: $\sqrt{x+2} > x$.
5. Solve for $x > 0$: $x^x \leq 4$.
6. Solve for x: $\sqrt{x+15} + \sqrt{19-x} < 8$.
7. Solve for x: $\frac{2x+5}{x^2+3x+2} \geq -\frac{1}{2}$.
8. Find all possible values for t such that $|t^2x^4 - (6t+2)x^2 + 9| \leq 8$ has no real solutions in x.
9. Given that $a, b, c > 1$ satisfies $a^{3a+2b+c} = b^{a+3b+2c} = c^{2a+b+3c}$, prove that $a = b = c$.

Hints...

1. Consider different cases for n: positive even, positive odd, 0, negative even, negative odd.

2. Complete the square.

3. $x^2 = |x|^2$.

4. Consider two cases before squaring both sides: $x \geq 0$ and $x < 0$. Also don't forget the restriction that $x + 2 \geq 0$.

5. Don't forget that the exponent rule depends on whether the base is less than or greater than 1, so you'll have to consider two cases for x: $x \leq 1$ and $x > 1$.

6. Square both sides, remembering the restrictions.

7. Rewrite the inequality as $\frac{P(x)}{Q(x)} \geq 0$ and factor $P(x)$ and $Q(x)$ to use property 11.

8. Get rid of the absolute value symbols using property 9. Note that the stuff in the absolute value symbols is a quadratic in x^2. If you have not learnt quadratic equations, this resource may be helpful. The Vieta's formula may help you, although it is not required to solve this question.

9. Note the symmetry of the equations. We can therefore, WLOG (witout loss of generality), assume that a is the minimum among a, b, c.