Answers and Brief Explanations to STOW #2: Logarithms

Cristian Bicheru

October 17, 2019

1 Logarithm Properties

Using exponent laws, one can derive the following properties:

$$\log_a(b^n) = n\log_a(b) \tag{1}$$
$$\log_a(b) + \log_a(c) = \log_a(bc) \tag{2}$$
$$\log_a(b) - \log_a(c) = \log_a\left(\frac{b}{c}\right) \tag{3}$$
$$\log_a(b) \cdot \log_c(d) = \log_a(d) \cdot \log_c(b) \tag{4}$$
$$\frac{\log_a(b)}{\log_a(c)} = \log_c(b) \tag{5}$$
$$\log_a(b) = \frac{1}{\log_b(a)} \tag{6}$$
$$\log_{a^n}(b^n) = \log_a(b) \tag{7}$$
$$\log_{\frac{1}{a}}(b) = -\log_a(b) \tag{8}$$

The proofs of these properties are left as an exercise to the reader. Logarithm properties are key to solving logarithm problems, so remember them well!

2 Practice Problems And Solutions

1. Find $\log_2(64)$.

   If we look at a few exponents of 2 we see that

$$2^1 = 2$$
$$2^2 = 4$$
$$2^3 = 8$$
$$2^4 = 16$$
$$2^5 = 32$$
$$2^6 = 64$$

   Therefore $\log_2(64) = 6$, since $2^6 = 64$.

2. Find $\log_3(27)$.

   Using the same method as in Problem 1, we see that $\log_3(27) = 3$.

3. Simplify $\frac{\log_2(16^{x^2-1})}{x+1}$.

   Using property (1), we get that

$$\begin{aligned}\frac{\log_2(16^{x^2-1})}{x+1} &= \log_2(16) \cdot \frac{x^2-1}{x+1}\\ &= \log_2(16) \cdot (x-1)\\ &= 4(x-1).\end{aligned}$$

4. Find the solution to $\log_{\frac{1}{2}}(3^x) \cdot \log_{\frac{1}{3}}(2^x) = 3$.

   Using property (4), we get that

$$\log_{\frac{1}{2}}(2^x) \cdot \log_{\frac{1}{3}}(3^x) = 3.$$

   Using property (1), we get that

$$x^2 \cdot \log_{\frac{1}{2}}(2) \cdot \log_{\frac{1}{3}}(3) = 3.$$
$$x^2 = 3$$
$$x = \pm 3.$$

5. Find the real solutions to $8 - \log_2(x) = 15\log_x(2)$.

Rearranging, we get that

$$15\log_x(2) + \log_2(x) - 8 = 0.$$

Using property (6), we get that

$$15\log_x(2) + \frac{1}{\log_x(2)} - 8 = 0.$$

Let $u = \log_x(2)$. Thus

$$15u + \frac{1}{u} - 8 = 0.$$

Multiplying by u, we get

$$15u^2 - 8u + 1 = 0$$

$$(5u - 1)(2u - 1) = 0$$

$$u = 5, 3$$

$$\log_x(2) = 5, 3$$

$$x = 32, 8$$

.

6. Find the real solution to $2 + \ln\sqrt{1+x} + 3\ln\sqrt{1-x} = \ln\sqrt{1-x^2}$.

Rearranging, we get that

$$2 = \ln\sqrt{1-x^2} - \ln\sqrt{1+x} - 3\ln\sqrt{1-x}.$$

Using property (1), we get that

$$2 = \ln\sqrt{1-x^2} - \ln\sqrt{1+x} - \ln((\sqrt{1-x}^3)).$$

Using property (3), we get that

$$2 = \ln\frac{\sqrt{1-x^2}}{\sqrt{1+x}(\sqrt{1-x})^3}$$

$$2 = \ln\frac{1}{(\sqrt{1-x})^2}$$

$$2 = \ln\frac{1}{1-x}$$

$$2 = \ln((1-x)^{-1}) = -\ln(1-x)$$

$$1 - x = \frac{1}{e^2}$$
$$x = 1 - \frac{1}{e^2}.$$

7. Find x if the triangle with side lengths $\ln(x^3), \ln(x^4), \ln(x^5)$ has area $15e^2$.

   Using property (1), we find that the side lengths form a Pythagorean Triple. The area is thus the opposite times the adjacent, which is $\ln(x^3) \cdot \ln(x^4)$. Setting this equal to $15e^2$, we get

$$15e^2 = 12\ln(x) \cdot \ln(x)$$
$$\frac{5}{4}e^2 = (\ln(x))^2$$
$$\ln(x) = \pm\frac{e\sqrt{5}}{2}$$
$$x = \exp\left(\pm\frac{e\sqrt{5}}{2}\right)$$

8. Evaluate $\prod_{i=2}^{2006} \log_i(i+1) = \log_2(3) \cdot \log_3(4) \cdots \log_{2005}(2006)$.

   Using property (4), we can cycle all the arguments of the log functions one over to the right

$$\prod_{i=2}^{2006} \log_i(i+1) = \log_2(3) \cdot \log_3(4) \cdots \log_{2005}(2006)$$
$$= \log_2(2006) \cdot \log_3(3) \cdots \log_{2005}(2005)$$
$$= \log_2(2006) \cdot 1 \cdots 1$$
$$= \log_2(2006).$$

9. Simplify $\frac{1}{\log_2(N)} + \frac{1}{\log_3(N)} + \frac{1}{\log_4(N)} + \cdots + \frac{1}{\log_{100}(N)}$ where $N = (100!)^3$.

   Using property (6), we get that

$$\frac{1}{\log_2(N)} + \frac{1}{\log_3(N)} + \frac{1}{\log_4(N)} + \cdot\cdot + \frac{1}{\log_{100}(N)} = \log_N(2) + \log_N(3) + \log_N(4) + \cdot\cdot + \log_N(100).$$

Using property (2), we get that

$$\log_N(2) + \log_N(3) + \log_N(4) + \cdots + \log_N(100) = \log_N(2 \cdot 3 \cdot 4 \cdots 100).$$
$$= \log_N(100!)$$
$$= \log_N(N^{\frac{1}{3}}).$$

Using property (1), we get that

$$\log_N(N^{\frac{1}{3}}) = \frac{1}{3} \cdot \log_N(N) = \frac{1}{3}.$$

10. Calculate the ratio $\frac{x}{y}$ if $2\log_5(x - 3y) = \log_5(2x) + \log_5(2y)$.

Using property (2), we get that

$$2\log_5(x - 3y) = \log_5(4xy).$$

Using property (1), we get that

$$\log_5((x - 3y)^2) = \log_5(4xy).$$

Raising both sides to the exponent 5, we get that

$$(x - 3y)^2 = 4xy$$
$$x^2 - 6xy + 9y^2 = 4xy$$
$$x^2 - 10xy + 9y^2 = 0$$
$$(x - 9y)(x - y) = 0.$$

Therefore either $x = y$ or $x = 9y$, which means $\frac{x}{y} = 1, 9$.

11. The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

Rewriting $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$ in terms of a and r, we get that

$$\log_8(a) + \log_8(ar) + \cdots + \log_8(ar^{11}) = 2006.$$

Using property (2), we get that

$$\log_8(a^{12}r^{66}) = 2006$$
$$2\log_8(a^6r^{33}) = 2006$$

$$\log_8(a^6r^{33}) = 1003$$
$$a^6r^{33} = 8^{1003} = 2^{1003^3}$$
$$(a^2r^{11})^3 = 2^{1003^3}$$
$$a^2r^{11} = 2^{1003}.$$

Since a and r must be positive integers, they must also be positive powers of two (since they multiply to a power of two). Let $a = 2^n$, $r = 2^m$. Therefore

$$(2^m)^2(2^n)^{11} = 2^{1003}$$
$$2^{2m} \cdot 2^{11n} = 2^{1003}$$
$$2^{2m+11n} = 2^{1003}$$
$$2m + 11n = 1003$$
$$n = \frac{1003 - 2m}{11}.$$

In order for a and r to be positive integers, m, n should be positive integers. Which means that $11|1003 - 2m$. If $m = 1$ this holds, because $1003 - 2 = 91 \cdot 11$. The next m that works is $m = 12$, because we are subtracting $2 \cdot 11 = 22$ from the numerator which is also divisible by 11. Thus $m = 1 + 11k$. The max m is 501, since after that n must be negative. Thus we must find how many values of $m = 1 + 11k$ are within the range [1, 501]. If $k = 45$, $m = 496$ but if $k = 46$, $m = 507$. Thus k can be any integer in the range [0, 45], which means there are 46 total possibilities.