STOW #2: Logarithms

Cristian Bicheru

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All of the fancy terms in this paper will have links to an appropriate definition. The links will also be on the Mathletes Classroom if you are too lazy to type it out.

1 Basic principles

1. **Logarithm Definition**

   The logarithm is defined as the inverse function to exponentiation. Basically, the logarithm of a number x is the exponent to which a base b is raised so that it produces the number x [1]. Obviously $b \neq 0, 1$ and $b < 0$ is outside the scope of this paper. An explicit way to write what we have just stated is:

   $$\log_b(x) = y \quad \text{iff (if and only if)} \quad b^y = x$$

   For example, since $2^6 = 64$, $\log_2(64) = 6$.

2. **Logarithm Conventions**

   Sometimes $\log(x)$ (with no base in front of the log) is used to refer to the logarithm base 10 of x, or $\log_{10}(x)$. Additionally, $\ln(x)$ refers to the logarithm base e (Euler's Number [2]) of x, or $\log_e(x)$.

3. **Logarithm Properties**

   Using exponent laws, one can derive the following properties [3]:

$$\log_a(b^n) = n\log_a(b) \tag{1}$$
$$\log_a(b) + \log_a(c) = \log_a(bc) \tag{2}$$
$$\log_a(b) - \log_a(c) = \log_a\left(\frac{b}{c}\right) \tag{3}$$
$$\log_a(b)\cdot\log_c(d) = \log_a(d)\cdot\log_c(b) \tag{4}$$
$$\frac{\log_a(b)}{\log_a(c)} = \log_c(b) \tag{5}$$
$$\log_a(b) = \frac{1}{\log_b(a)} \tag{6}$$
$$\log_{a^n}(b^n) = \log_a(b) \tag{7}$$
$$\log_{\frac{1}{a}}(b) = -\log_a(b) \tag{8}$$

   The proofs of these properties are left as an exercise to the reader. Logarithm properties are key to solving logarithm problems, so remember them well!

2 Practice Problems

1. Find $\log_2(64)$.
2. Find $\log_3(27)$.
3. Simplify $\frac{\log_2(16^{x^2-1})}{x+1}$.
4. Find the solution to $\log_{\frac{1}{2}}(3^x)\cdot\log_{\frac{1}{3}}(2^x) = 3$.
5. Find the real solutions to $8 - \log_2(x) = 15\log_x(2)$.
6. Find the real solution to $2 + \ln\sqrt{1+x} + 3\ln\sqrt{1-x} = \ln\sqrt{1-x^2}$.
7. Find x if the triangle with side lengths $\ln(x^3), \ln(x^4), \ln(x^5)$ has area $15e^2$.
8. Evaluate $\displaystyle\prod_{i=2}^{2005}\log_i(i+1) = \log_2(3)\cdot\log_3(4)\cdots\log_{2005}(2006)$.
9. Simplify $\frac{1}{\log_2(N)} + \frac{1}{\log_3(N)} + \frac{1}{\log_4(N)} + \cdots + \frac{1}{\log_{100}(N)}$ where $N = (100!)^3$.
10. Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$.
11. The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

References

[1] Wikipedia: Logarithm,
https://en.wikipedia.org/wiki/Logarithm

[2] Wikipedia: e (mathematical constant),
https://en.wikipedia.org/wiki/E_(mathematical_constant)

[3] AoPS: Logarithm,
https://artofproblemsolving.com/wiki/index.php/Logarithm