JTOW #2: Completing the Square

Zed Li

October 11, 2019

1 Basic principles

1. *Completing the square* involves factoring (a part of) an expression using the **square of sum formula:**
$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$
or in general
$$\left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^2 + 2\sum_{i<j} x_i x_j$$
i.e.
$$\begin{aligned}(x_1 + x_2 + \cdots x_n)^2 = &\, x_1^2 + x_2^2 + \cdots + x_n^2 \\ &+ 2x_1x_2 + 2x_1x_3 + \cdots + 2x_1x_n \\ &+ 2x_2x_3 + 2x_2x_4 + \cdots + 2x_2x_n \\ &+ \cdots \\ &+ 2x_{n-1}x_n.\end{aligned}$$

2. **Important property:**
$$x^2 \geq 0 \tag{1}$$
for all real x. Equality holds if and only if $x = 0$.

3. **Example applications:**

   (a) *Reducing the order of a polynomial.* As an example, after you get an expression squared on both sides, you can take the square root on both sides, you half the degree of the polynomial. This is how a quadratic formula is derived: by reducing a quadratic equation to a linear one.

   (b) *Solving an equation with more than one variable in the reals or in the integers.* For example, if an equation can be written as the sum of the squares of several expressions equaling zero, then each individual expression equals zero. (Try to prove this yourself using Equation 1; *Hint:* Proof by contradiction.)

   (c) Oftentimes, rewriting an expression by completing the square might help in general.

2 Practice problems

See hints on the next page.

1. Given $x + \frac{1}{x} = 3$, find $x^4 + \frac{1}{x^4}$ without solving for x using the quadratic formula.
2. Solve $a^2 + b^2 - 4a + 4b + 8 = 0$ for real numbers a, b.
3. Solve $x^2 - xy + y^2 + x + y + 1 = 0$ for real numbers x, y.
4. Solve $3a^2 + 3b^2 + 3c^2 - 2bc - 2ca - 2ab - 4 = 0$ for integers a, b, c.
5. Solve $t^4 - 7t^3 + 14t^2 - 7t + 1 = 0$ for real number t.

Haha, trolled! No hints!

But seriously, you should try to work on each question for long enough (e.g. at least 10 or 20 minutes) until you look at the hints. Also, discuss with your friends before taking a peek at the hints.

Hints...

1. Use the square of sum formula on $x + \frac{1}{x}$ and then $x^2 + \frac{1}{x^2}$.
2. Rewrite 8 as $4 + 4$.
3. Multiply both sides of the equation by 2 so that you can split terms apart like in the hint to question 2.
4. Consider the expansion of $(a + b - c)^2$.
5. Divide both sides by t^2, and refer to the trick in question 1.