Answers and Brief Explanations to STOW #1: Factoring

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1. $2(x+3)^2$

2. $(x^2+1)(x+1)(x-1)$

3. $(2a-3b)(4a^2+6ab+9b^2)$

4. $(3x+7y)(x^2+y^2)$ combine like terms

5. $(x-2y-3)((x-3)^4+2(x-3)^3y+4(x-3)^2y^2+8(x-3)y^3+16y^4)$ factor it to the exponent of sum and apply the difference of exponents

6. $(p+5)(p-5)(q+3)$ factor by grouping

7. $(c+4)(c^3+c^2+3c+1)$ apply rational root and factor theorems

8. $a^2+b^2+c^2+2ab+2bc+2ca=(a+b+c)^2$

9. i) $x^4+x^2+1=(x^2+x+1)(x^2-x+1)$ (break x^2 into $2x^2-x^2$ then group into perfect squares and do difference of squares or consider primitive 3rd roots of unity ω and ω^2)

   ii) $x^4+x^2+1=(x-\omega)(x+\omega)(x-\omega^2)(x+\omega^2)$ where $\omega=\frac{-1+\sqrt{3}i}{2}, \omega^2=\frac{-1-\sqrt{3}i}{2}$

10. $a^4+b^4=(a^2+\sqrt{2}ab+b^2)(a^2-\sqrt{2}ab+b^2)$ (break 0 into $2a^2b^2-2a^2b^2$ and group into perfect squares and do difference of squares)

11. $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ (observe that $a=-b-c$ is a root so according to the factor theorem, $a+b+c$ is a factor; the other factor must then be a homogeneous cyclic quadratic in three variables, namely of the form $k(a^2+b^2+c^2)+l(ab+bc+ca)$; then find k, l by comparing coefficients)

12. $\prod_{t=0}^{k-1}\left(n^{2^t}+1\right)$ (the original thingy is equal to $\frac{n^{2^k}-1}{n-1}$; do difference of squares over and over again and then cancel out $n-1$ in numerator and denominator)

13. i) $x^{12}+x^9+x^6+x^3+1=(x^4+x^3+x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1)$ (note that all primitive 5th roots of unity are roots of that expression, so $x^4+x^3+x^2+x+1$ is a factor; the rest is just breaking like terms and grouping, or equivalently long division)

    ii) $x^{12}+x^9+x^6+x^3+1=\prod_{k\in A}\left(x^2-2\cos\left(\frac{2k\pi i}{15}\right)x+1\right)$ where $A=\{1,2,3,4,6,7\}$ (product of all 15th roots of unity except for 3rd roots of unity)

    iii) $x^{12}+x^9+x^6+x^3+1=\prod_{k\in B}\left(x-e^{\frac{2k\pi i}{15}}\right)$ where $B=\{1,2,3,4,6,7,8,9,11,12,13,14\}$

14. i) $x^4+x^3-5x^2+8x-2=(x^2-2x+2)(x^2+3x-1)$ (rip, gotta solve for the coefficients)

    ii) $x^4+x^3-5x^2+8x-2=\left(x+\frac{3}{2}+\frac{\sqrt{13}}{2}\right)\left(x+\frac{3}{2}-\frac{\sqrt{13}}{2}\right)(x-1+i)(x-1-i)$ (have u ever heard of the qUAdRAtiC fORmulA?)

15. $-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2c^2a^2=(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$ (write $2a^2b^2$ as $-2a^2b^2+4a^2b^2$; or, if you're really good at expanding, you may note that $a=-b-c$ and $a=b+c$ are roots, so $a+b+c$ and $-a+b+c$ are factors, and then from symmetry $a-b+c$ and $a+b-c$ are factors as well, and the only factor left must be a constant, which is determined to be 1 from comparing the coefficients)

16. $a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc=(a+b)(b+c)(c+a)$ (if you're good, break apart $2abc=abc+abc$ and then do grouping to make common factor of $a+b$ appear, and then you get another factor of $ab+bc+ca+c^2$ which can again be factored by grouping; but why torture yourself? just do factor theorem observing that $a=-b$ is a root and by symmetry $a+b, b+c, c+a$ are all factors and at last determine the constant coefficient which is 1)

17. $a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc=(a+b+c)(bc+ca+ab)$ (if you're good, break apart $3abc=abc+abc+abc$ and then do grouping to make common factor of $bc+ca+ab$ appear, and then you common factor; but why torture yourself? just do factor theorem observing that $a=-b-c$ is a root so $a+b+c$ is a factor; the other factor is then homogeneous, cyclic and of degree 2, i.e. it's of the form $k(a^2+b^2+c^2)+l(bc+ca+ab)$, so you can compare coefficients to get $k=0$ and $l=1$)