J/STOW #1: Factoring

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October 4, 2019

If you don't know what some stuff in this pdf means (e.g. *imaginary/complex numbers*, *homogeneous polynomials*)

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hover over the link
or search it up
or ask someone

1 Basic principles

1. **Common factoring** i.e. distributive property of multiplication: $ab_1 + ab_2 + \cdots + ab_n = a(b_1 + b_2 + \cdots + b_n)$ (you should know this from math class).

2. **Using formulas**

   - $(x+y)^2 = x^2 + 2xy + y^2$ (I really hate writing $(x \pm y)^2 = x^2 \pm 2xy + y^2$ b/c that's really saying the exact same thing but you waste ink writing two extra horizontal lines.)
   - $x^2 - y^2 = (x-y)(x+y)$
   - $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ (general case for formula 1) where $\binom{n}{k}$ is the binomial coefficient function.
   - $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$ (Again, I don't like wasting ink writing $x^3 \mp y^3 = (x-y)(x^2 \pm xy + y^2)$.)
   - $x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k} y^k$ (general case for formulas 2 and 4)

   Try proving these formulas rigorously (long division would not be accepted as a rigorous proof for formula 5).

3. **Group the terms** When there is no common factor, put the terms into several groups, each of which contains terms with a common factor. Common factor each group, and hope that you can proceed the factoring from the result, e.g. $ax + ay + bx + by = a(x+y) + b(x+y) = (a+b)(x+y)$

4. **Breaking apart like terms** Write a term as two or more like terms that add to that term (e.g. write 0 as $x - x$ or $2a$ as $a + a$) and then group the terms, e.g. $x^3 + x^2 + 4 = (x^3 + 2x^2) - (x^2 + 2x) + (2x + 4) = x^2(x+2) - x(x+2) + 2(x+2) = (x+2)(x^2 - x + 2)$

- Quadratics with one variable or homogeneous quadratic with two variables (e.g. $x^2+4x-21$ or $x^2+4xy-21y^2$): use the chart thingy taught in class for normal single-variable quadratics. Basically, for $ax^2+bxy+cy^2$, you want to find $p, q.r, s$ such that $pq=a$, $ps+qr=b$ and $rs=c$. Then $ax^2+bxy+cy^2=pqx^2+(ps+qr)xy+rsy^2=px(qx+sy)+ry(qx+sy)=(px+ry)(qx+sy)$.

5. **Rational root theorem and factor theorem** Consider the polynomial with integer coefficients and $a_n \neq 0$: $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$. Any rational root of $P(x)$ can be written of the form $\frac{p}{q}$ $(p, q \in \mathbb{Z}, q \neq 0)$ where p is a factor of a_0 and q is a factor of a_n. (*Try to prove this yourself rigorously.*) Then, $qx-p$ is a factor of $P(x)$.

6. **Symmetry** If polynomial $P(x_1, x_2, \ldots, x_{n-1}, x_n)$ is symmetrical in a way such that $P(x_2, x_3, \ldots, x_n, x_1) \equiv P(x_1, x_2, \ldots, x_{n-1}, x_n)$, then if polynomial $D(x_1, x_2, \ldots, x_{n-1}, x_n)$ is a factor, $D(x_2, x_3, \ldots, x_n, x_1)$ is also a factor.

7. **Roots of unity** The polynomial $P_n(x)=x^n-1$ has n roots, namely $x_k=e^{\frac{2\pi ki}{n}}=\cos\left(\frac{2\pi ki}{n}\right)+i\sin\left(\frac{2\pi ki}{n}\right)$ for $k=0,1,2,\ldots,n-1$. We call these roots *roots of unity* since they lie on the unit circle. (See link if you don't know Euler's formula.)

   - Allowing complex coefficients, $P_n(x)=\prod_{k=0}^{n-1}\left(x-e^{\frac{2\pi ki}{n}}\right)$. (See link for capital Pi notation.)
   - Allowing real coefficients, note that in the expression right above we can multiply each pair of two factors corresponding to a conjugate pair of complex roots to create a quadratic with real coefficients. That is, $\left(x-e^{\frac{2\pi ki}{n}}\right)\left(x-e^{-\frac{2\pi ki}{n}}\right)=x^2-2\cos\left(\frac{2\pi ki}{n}\right)x+1$.
   - Allowing rational coefficients, one can consider the factors corresponding to primitive roots of unity of $P_k(x)$ for each k being a positive factor of n. These factors multiplied out would result in a factor of $P_n(x)$ that has rational coefficients.

8. **Solve for coefficients** e.g. when you have a quartic that factors into two quadratics that cannot be further factored (allowing only rationals coefficients), let it equal to $(ax^2+bx+c)(dx^2+ex+f)$ and solve for the coefficients a, b, c, d, e, f.

2 Practice problems

Note that when I say "in the rationals/reals/complex," I mean the coefficients of each of the factors are restricted to rational/real/complex numbers; I'm not sure if this is a conventional way of saying it, but whatever. (FYI in math class we only learn to factor in the rationals)

Factor the following completely: *(Note: Questions 1 to 7 focus on the fundamentals and are all to be factored in the rationals; questions 8 to 17 focus on the challenge.)*

1. $2x^2+12x+18$
2. x^4-1

3. $8a^3 - 27b^3$
4. $8x^3 - 12xy^2 - 3y^3 + 23x^2y + 10y^3 - 5x^3 - 16x^2y + 15xy^2$
5. $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243 - 32y^5$
6. $p^2q - 25q + 3p^2 - 75$
7. $c^4 + 5c^3 + 7c^2 + 11c + 4$
8. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ in the rationals
9. $x^4 + x^2 + 1$ in the i) rationals, ii) complex
10. $a^4 + b^4$ in the reals (Thought this was a sum of squares and was not factorable from math class?? This will blow your mind! In fact, all univariate polynomials or bivariate homogeneous polynomials can be factored into linears and quadratics in the reals!!)
11. $a^3 + b^3 + c^3 - 3abc$ in the rationals
12. $\sum_{t=0}^{2^k-1} n^t$ in the rationals (see link for capital-sigma notation)
13. $x^{12} + x^9 + x^6 + x^3 + 1$ in the i) rationals, ii) reals, iii) complex
14. $x^4 + x^3 - 5x^2 + 8x - 2$ in i) the rationals ii) the complex
15. $-a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2$ in the rationals
16. $a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc$ in the rationals
17. $a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc$ in the rationals